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COMPLEX SUPERPOSITION ALGEBRA

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ABSTRACT

We will use the notions of real axis and imaginary axis to construct by orthogonal superposition a complex plan with complex base of module $\rho = \sqrt{1} = 1$ that is analogous to the set C of complex numbers. Then we will superimpose the three complex planes to obtain the hyper-space of six (06) dimensions. We will thus arrive at the algebra of six (06) dimension: three (03) real dimensions and three (03) imaginary dimensions. The construction of the algebra requires developing and defining a commutative multiplication table using the exponential notations of Euler's formula. What we will generalize, by defining new multiplication rules, by to the notions of "the real product" and "imaginary product"; which we will call "Euler products". This superposition reveals two new complex bases with different imaginary modules: $\sqrt{2}$; and $\sqrt{3}$ in the hyper-space of six dimensions. What is magical about this multiplication, is the impression we have of jumping from one orbital to another.

1 INTRODUCTION

This text is a geometric approach to construct a commutative, associative, bilinear and unitary algebraic structure of "superimposed hyper-complex" numbers for dimensions $n \in \mathbb{N}$. We know that the set $\mathbb{C}$ of complex numbers is defined as an extension of the set $\mathbb{R}$ of real numbers containing an imaginary number denoted i ; such that: $i^2 = -1$. Any complex number s can then be written in the form: $s = x + iy$ (where x and y are real numbers).
Complex numbers were gradually introduced in the 16th century by the Italian mathematical school (Jérôme Cardan, Raphaël Bombelli, Tartaglia…) in order to express the solutions of third degree equations in complete generality by Cardan's formulas, in using in particular negative square "numbers". We can provide the set of complex numbers with an addition and a multiplication, which make it a commutative field containing the field of real numbers.
In 1806, while running a bookstore in Paris, the Swiss Jean Robert Argand published a geometric interpretation of complex numbers as points in the plane, by corresponding the complex number $a + b\sqrt{-1}$; to the unique point of the coordinate plane (a; b). For this reason, the plane, seen as a set of complex numbers, is sometimes called the Argand plane. Also the Frenchman Jacques Fréderic Français, who discovered the work of Argand adds that; geometrically the imaginary number $\sqrt{-1}$ is the image in the complex plane (Argand plane) of the real number 1 by the rotation with center O and angle $\theta = \frac{\pi}{2}$ and note $\sqrt{-1} = 1_{\frac{\pi}{2}}$. To this must be added the publications of Abbé Buée, the Danish and Norwegian Caspar Wessel and others in the development of the geometric aspect of complex numbers.
However, this geometric conception of an algebraic tool clashes with the logical sense of certain mathematicians of the time. It was only when Gauss and de Cauchy took up this idea that this conception acquired its letters of nobility. Thus, the geometric aspect of complex numbers develops; they are associated with vectors or points on the plane. The transformations of the plan are then expressed in the form of complex transformations. Which allowed William Rowan Hamilton to create his quaternions. In 1843 Hamilton who was the first to give the algebraic writing of a hyper-complex number (quaternions) in the $q = x + iy + jz + kt$ (where x ; y ; z and t are real numbers and i ; j and k are imaginary numbers) ; with one of the conditions $i^2 = j^2 = k^2 = -1$.
In mathematics, the term hyper-complex number is used to refer to elements of algebras that are extended or go deeper than complex number arithmetic. An accessible and modern definition of a hyper-complex number is given by Kantor and Solodovnikov. They are elements of a unitary real algebra (not necessarily associative) of dimension $n + 1 > 0$.
Hyper-complex numbers are obtained by further generalizing the construction of complex numbers from real numbers by the Cayley-Dickson construction. This allows complex numbers to be extended into algebras of dimension 2^n $(n \in \{2 ; 3 ; 4 \dots \dots\})$. The best known are the algebra of quaternions (of dimension 4), that of octonions (of dimension 8) and that of sedenions (of dimension 16). However, increasing the dimension introduces algebraic complications: the multiplication of quaternions is no longer commutative, the multiplication of octonions is, moreover, non-associative and the norm on sedenions is not multiplicative. In the definition of Kantor and Solodovnikov, these numbers correspond to anti-commutative bases of type $i_m^2 = -1$ (avec $m \in \{1 ; \dots \dots ; 2^n - 1\}$). Since quaternions and octonions provide a similar (multiplicative) norm to the lengths of Euclidean vector spaces of dimensions four and eight respectively, they can be associated with points in some higher-dimensional Euclidean spaces. Beyond the octonions, on the other hand, this analogy falls away since these constructions are no longer normed.
We will present here a geometric approach by orthogonal superposition of complex plans (with real and imaginary bases). Which, as we will see, will have the consequence of giving a commutative, associative, bilinear and unitary algebraic structure to the set of numbers that we will call here "superimposed hyper-complex" numbers.
I would present the construction methods by geometric superposition, which result in a commutative algebra of "superimposed hyper-complex" numbers. In addition, the introduction of new concepts such as; "Euler bases" obtained using exponential notation of Euler's formula; the "real product" and the "imaginary product"; allowing multiplication to remain commutative. Which gives rise to new complex bases, in which the multiplication passes from one base to another in commutative way. These bases have different modules and are distributed in a discreet manner. Above all, we will see that the hyper-complex writing $q = x + iy + jz + kt$ of quaternions defined by Hamilton, associated with the "real product" and the "imaginary product", describes a state of superposition.

2 SET $\mathbb{S}_{(1;\,1)}$

2.1 Real axis – Imaginary axis

Consider a line $(d_{n,\propto})$ provided with a coordinate system $(O ;\ v_{n,\propto})$ with $n \in \{1 ; 2 ; 3 ; \dots \dots \dots\}$ and $\propto \in \{r ; i\}$ where r denotes the word "real" and i denotes the word "imaginary". We can associate with this axis a real graduation unit $|v_{n,r}| = 1$ or an imaginary graduation unit $|v_{n,i}| = 1$; defined by:

- $v_{n,r}{}^2 = v_{n,r}$ if $v_{n,r}$ is real.

- ${v_{n,i}}^2 = -v_{n,r}$ if $v_{n,i}$ is imaginary.

Thus any point M belonging to $(d_{n,\propto})$ has for affix the number h_M such that: $h_M = a_{n,\propto} v_{n,\propto}$
Thus h_M is a linear combination with real coefficient $a_{n,\propto} \in \mathbb{R}$ and a canonical basis $\{v_{n,\propto}\}$; (with $n \in \{1; 2; 3; \dots\}$; $\propto \in \{r ; i\}$).

2.2 Algebraic space $\mathbb{S}_{1,r}$ dimension 1 – Vector space $\mathbb{S}_{1,i}$ dimension 1

We thus define the set $\mathbb{S}_{n,\propto}$ the set of numbers h affixes of the points M of a line $(d_{n,\propto})$ and which are written in the form: $h = a_{n,\propto} v_{n,\propto}$ (Where $a_{n,\propto} \in \mathbb{R}$ and $v_{n,\propto}$ being a real or imaginary one).

2.2.1 Algebraic space $\mathbb{S}_{1,r}$

An algebraic space denoted $\mathbb{S}_{1,r}$ such that any real number $h \in \mathbb{S}_{1,r}$ is written $h = x_{1,r} v_{1,r}$ (where $v_{1,r}$ is real and $x_{1,r} \in \mathbb{R}$). Under these conditions, we can also show that $\mathbb{S}_{1,r}$ is an $\mathbb{R}$-Algebra of dimension 1, provided with the internal operations $+$ and $\times$ from $\mathbb{S}_{1,r}$ to $\mathbb{S}_{1,r}$ defined by:
If $v_{1,r}$ is a real base we have:

$$\begin{cases} h + h' = x_{1,r} v_{1,r} + {x_{1,r}}' v_{1,r} = (x_{1,r} + {x_{1,r}}') v_{1,r} \\ h \times h' = x_{1,r} v_{1,r} \times {x_{1,r}}' v_{1,r} = (x_{1,r} \times {x_{1,r}}') {v_{1,r}}^2 = x_{1,r} {x_{1,r}}' v_{1,r} \end{cases}$$

Since the addition of real numbers is commutative, associative and admits 0 as a neutral element and just as the multiplication of real numbers is commutative, associative, distributive with respect to addition and admits 1 as a neutral element then we have:

- $h + h' = h' + h$
- $h + 0_{\mathbb{S}} = h$
- $h \times h' = h' \times h$
- $v_{1,r} \times h = v_{1,r} \times (x_{1,r} v_{1,r}) = x_{1,r} v_{1,r} \times v_{1,r} = x_{1,r} {v_{1,r}}^2 = x_{1,r} v_{1,r} = h$

$\mathbb{S}_{1,r}$ is an associative, commutative bilinear and unitary algebra on $\mathbb{R}$.

2.2.2 The vector space $\mathbb{S}_{1,i}$

The vector space denoted $\mathbb{S}_{1,i}$ is such that any imaginary number $h \in \mathbb{S}_{1,i}$ is written $h = y_{1,i} v_{1,i}$ (where $v_{1,i}$ is an imaginary basis and $y_{1,i} \in \mathbb{R}$). Under these conditions, we can show that the set $\mathbb{S}_{1,i}$ is a vector R-Space equipped with the internal operation + from $\mathbb{S}_{1,i}$ to $\mathbb{S}_{1,i}$ defined by:
If $v_{1,i}$ is an imaginary basis, we have: $h + h' = y_{1,i} v_{1,i} + {y_{1,i}}' v_{1,i} = (y_{1,i} + {y_{1,i}}') v_{1,i}$
Since the addition of real coefficients is commutative, associative and admits 0 as a neutral element then the internal operation $+$ from $\mathbb{S}_{1,i}$ to $\mathbb{S}_{1,i}$ verifies the following axioms:

- $h + h' = h' + h$
- $(h + h') + h'' = h + (h' + h'')$
- $h + 0_{\mathbb{S}} = h$
- If $h + h' = 0_{\mathbb{S}}$ then $h' = -h$

Likewise, the multiplication of real numbers being commutative, associative, distributive with respect to addition and admits 1 as a neutral element, then the external operation $\times$ from $\mathbb{S}_{1,i}$ to $\mathbb{S}_{1,i}$ verifies the following axioms:
If $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$ we have:

- $1 \times h = h$
- $\lambda(\mu h) = (\lambda\mu) h$
- $\lambda(h + h') = \lambda h + \lambda h'$
- $(\lambda + \mu) h = \lambda h + \mu h$
- $h \times 0 = 0_{\mathbb{S}}$

Which gives $\mathbb{S}_{1,i}$ a vector space structure on $\mathbb{R}$.

2.3 Hyper-complex algebraic space $\mathbb{S}_{(1;\,1)}$

2.3.1 Hyper-complex basis $\{v_{1,r}\ ;\ v_{2,i}\}$

We can define a hyper-complex plane $P_{(1;1)}$ as being a plane formed by two perpendicular axes: an axis with a real base $v_{1,r}$ and an imaginary base axis $v_{2,i}$. We associate with a hyper-complex plan a direct direction of rotation from the real axis towards the imaginary axis. If we consider the hyper-complex basis $\{v_{1,r}\ ;\ v_{2,i}\}$ defined by:

$$\begin{cases} 1_{(v_{1,r}\,;\,v_{2,i})} = v_{1,r} \\ {v_{1,r}}^2 = v_{1,r} \\ v_{2,i}^2 = -v_{1,r} \\ v_{1,r} \times v_{2,i} = v_{2,i} \times v_{1,r} = v_{2,i} \end{cases}$$

We can associate at any point of the hyper-complex plane $(O; v_{1,r}\ ;\ v_{2,i})$; a hyper-complex number h. This number being the sum of a number h_r affix of a point on the real line $(d_{1,r})$ and of a number h_i affix of a point on the imaginary line $(d_{2,i})$; such as:
$h = h_r + h_i$ where $h_r \in \mathbb{S}_{1,r}$ and $h_i \in \mathbb{S}_{1,i}$.
We then have: $h = x_{1,r}v_{1,r} + y_{2,i}v_{2,i}$
We can add that the hyper-complex number h of the hyper-complex base $\{v_{1,r}\ ;\ v_{2,i}\}$ can be written in several other forms:

- $h = \rho\left(v_{1,r}\cos\theta + v_{2,i}\sin\theta\right)$ with $\theta \in \mathbb{R}$ such that: $\begin{cases} \rho = \sqrt{{x_1}^2 + {y_2}^2} \\ \cos\theta = \frac{x_1}{\sqrt{{x_1}^2+{y_2}^2}} \\ \sin\theta = \frac{y_2}{\sqrt{{x_1}^2+{y_2}^2}} \end{cases}$
- $h = \rho e^{\theta v_{2,i}}$ with $\theta \in \mathbb{R}$

We can demonstrate it using the Taylor series that: $e^{\theta v_{2,i}} = v_{1,r}\cos\theta + v_{2,i}\sin\theta$ with $\theta \in \mathbb{R}$.
The Taylor series expansion of the exponential function of the real variable t can be written:

$$e^t = \frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{t^n}{n!}$$

In the hyper-complex basis $\{v_{1,r}\ ;\ v_{2,i}\}$; in particular for $t = \theta v_{2,i}$ with real θ, we have:

$$e^{\theta v_{2,i}} = \sum_{n=0}^{\infty}\frac{\left(\theta v_{2,i}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{{v_{2,i}}^n\,\theta^n}{n!}$$

This series can be separated into two by grouping the even and odd terms. Indeed, a rearrangement of the order of the terms of the series is possible here, because it is an absolutely convergent series, in other words a summable family. We then obtain, using the fact that:
${v_{2,i}}^{2k} = \left({v_{2,i}}^2\right)^k = \left(-v_{1,r}\right)^k$ and ${v_{2,i}}^{2k+1} = {v_{2,i}}^{2k}v_{2,i} = \left(-v_{1,r}\right)^k v_{2,i}$
So

$$e^{\theta v_{2,i}} = \sum_{k=0}^{\infty}\frac{{v_{2,i}}^{2k}\theta^{2k}}{2k!} + \sum_{k=0}^{\infty}\frac{{v_{2,i}}^{2k+1}\theta^{2k+1}}{(2k+1)!}$$

$$e^{\theta v_{2,i}} = \sum_{k=0}^{\infty}\frac{\left(-v_{1,r}\right)^k\theta^{2k}}{2k!} + \sum_{k=0}^{\infty}\frac{\left(-v_{1,r}\right)^k v_{2,i}\,\theta^{2k+1}}{(2k+1)!}$$

$$e^{\theta v_{2,i}} = \sum_{k=0}^{\infty}\frac{(-1)^k\left(v_{1,r}\right)\theta^{2k}}{2k!} + \sum_{k=0}^{\infty}\frac{(-1)^k v_{1,r}v_{2i}\,\theta^{2k+1}}{(2k+1)!}$$

$$e^{\theta v_{2,i}} = v_{1,r}\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{2k!} + v_{1,r}v_{2,i}\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}$$

$$e^{\theta v_{2,i}} = v_{1,r}\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{2k!} + v_{2,i}\sum_{k=0}^{\infty}\frac{(-1)^k\,\theta^{2k+1}}{(2k+1)!}$$

We thus see the Taylor series expansions of the cosine and sine functions appear:

$$\cos\theta = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{2k!} \quad \text{and} \quad \sin\theta = \sum_{k=0}^{\infty}\frac{(-1)^k\,\theta^{2k+1}}{(2k+1)!}$$

Which, by replacing $e^{\theta v_{2,i}}$, in the previous expression, gives in the hyper-complex basis $\{v_{1,r}\ ;\ v_{2,i}\}$: $e^{\theta v_{2,i}} = v_{1,r}\cos\theta + v_{2,i}\sin\theta$
So $\mathrm{h} = \rho\left(v_{1,r}\cos\theta + v_{2,i}\sin\theta\right) \iff \mathrm{h} = \rho e^{\theta v_{2,i}}$

2.3.2 Euler identity and "Euler basis"

According to the equality $e^{\theta v_{2,i}} = v_{1,r}\cos\theta + v_{2,i}\sin\theta$ we have:

- $e^{2k\pi v_{2,i}} = v_{1,r}$ if $\theta = 2k\pi$ (we take $k = 0$)
- $e^{\frac{\pi}{2}v_{2,i}} = v_{2,i}$ if $\theta = \frac{\pi}{2}$
- $e^{\pi v_{2,i}} = -v_{1,r}$ if $\theta = \pi$

The Euler identity in this basis becomes $e^{\pi v_{2,i}} + v_{1,r} = 0$

By definition, we call the Euler base the base $\{v_{1,r}\ ;\ v_{2,i}\}$; such as: $\begin{cases} v_{1,r} = e^{2k\pi v_{2,i}} \\ v_{2,i} = e^{\frac{\pi}{2} v_{2,i}} \end{cases}$

So if $v_{1,r} = 1$ and $v_{2,i} = i$ then $\mathbb{S}_{(1;1)} = \mathbb{C}$

2.3.3 Direct similarity matrix of a hyper-complex number

For a hyper-complex number $\mathrm{h} = x_{1,r}v_{1,r} + y_{2,r}v_{2,i}$ of dimension 2 its corresponding direct similarity matrix is written as follows:

$$\mathcal{M}_2(h) = \begin{pmatrix} x_{1,r} & -y_{2,i} \\ y_{2,i} & x_{1,r} \end{pmatrix}$$

2.4 Operations in $\mathbb{S}_{(1\ ;\ 1)}$

By the following steps we will show that the set $\mathbb{S}_{(1;1)}$ is an associative, commutative, bilinear and unitary algebra on the field of real numbers $\mathbb{R}$ in the hyper-complex base $\{v_{1,r}\ ;\ v_{2,i}\}$

According to the equalities: $\mathrm{h} = \rho\left(v_{1,r}\cos\theta + v_{2,i}\sin\theta\right)$ whith $\theta \in \mathbb{R}$

We can clearly see that the rules of addition and multiplication in $\mathbb{S}_{(1;1)}$ are the same as those in $\mathbb{C}$.

2.4.1 Addition (internal operation $+$)

The addition of two hyper-complex numbers of dimension 2 in the hyper-complex basis $\{v_{1,r}\ ;\ v_{2,i}\}$; $h = x_{1,r}v_{1,r} + y_{2,i}v_{2,i}$ and $h' = x_{1,r}'v_{1,r} + y_{2,i}'v_{2,i}$; are then defined by:

$h + h' = \left(x_{1,r}v_{1,r} + y_{2,i}v_{2,i}\right) + \left(x_{1,r}'v_{1,r} + y_{2,i}'v_{2,i}\right)$

$h + h' = \left(x_{1,r} + x_{1,r}'\right)v_{1,r} + \left(y_{2,i} + y_{2,i}'\right)v_{2,i}$

We know that the addition of real coefficients is commutative, associative and admits 0 as a neutral element; we deduce that internal operation $+$ (addition) in $\mathbb{S}_{(1;1)}$ is a commutative, associative operation and admits $0_{\mathbb{S}}$ as a neutral element.

2.4.2 Multiplication (internal operation $\times$)

Similarly, the multiplication of two hyper-complex numbers of dimension 2 in the hyper-complex base $\{v_{1,r}\ ;\ v_{2,i}\}$ is then defined by the table:

$\times$	$v_{1,r}$	$v_{2,i}$
$v_{1,r}$	$v_{1,r}$	$v_{2,i}$
$v_{2,i}$	$v_{2,i}$	$-v_{1,r}$

Table 1: multiplication table of base $\{v_{1,r}\ ;\ v_{2,i}\}$

We thus obtain:

$h_1 \times h_2 = \left(x_{1,r}v_{1,r} + y_{2,i}v_{2,i}\right) \times \left(x_{1,r}'v_{1,r} + y_{2,i}'v_{2,i}\right)$

$h_1 \times h_2 = \left(x_{1,r}x_{1,r}' - y_{2,i}y_{2,i}'\right)v_{1,r} + \left(x_{1,r}y_{2,i}' + x_{1,r}'y_{2,i}\right)v_{2,i}$

Likewise we know that the multiplication of real coefficients is commutative, associative, distributive with respect to addition and admits 1 as a neutral element; thus we deduce that the internal operation $\times$ (multiplication) in $\mathbb{S}_{(1;1)}$ is a commutative, associative, bilinear operation and admits $v_{1,r}$ as a neutral element.

We can conclude that $\mathbb{S}_{(1;1)}$ has the internal operations $+$ and $\times$; has the same properties as those in $\mathbb{C}$. Thus $\mathbb{S}_{(1;1)}$) is an associative, commutative, unitary and bilinear algebra on the set of real numbers $\mathbb{R}$.

2.4.3 Hyper-complex bases $\{v_{2,r}\ ;\ v_{3,i}\}$ and $\{v_{3,r}\ ;\ v_{1,i}\}$

By the same approach we will show that the set $\mathbb{S}_{(1;1)}$ associated with the bases $\{v_{2,r}\ ;\ v_{3,i}\}$ and $\{v_{3,r}\ ;\ v_{1,i}\}$ is an associative, commutative, unitary and bilinear algebra on the field $\mathbb{R}$ of real numbers:

- **Euler basis $\{v_{2,r}\ ;\ v_{3,i}\}$**

The hyper-complex basis $\{v_{2,r}\ ;\ v_{3,i}\}$; is defined as follows:

- $e^{\theta v_{3,i}} = v_{2,r}\cos\theta + v_{3,i}\sin\theta$
- $e^{2k\pi v_{3,i}} = v_{2,r}$ we take $k = 0$
- $e^{\frac{\pi}{2}v_{3,i}} = v_{3,i}$
- $e^{\pi v_{3,i}} = -v_{2,r}$

The Euler identity in the base $\{v_{2,r}\ ;\ v_{3,i}\}$ becomes $e^{\pi v_{3,i}} + v_{2,r} = 0$

Likewise we obtain its Euler base $\{v_{2,r}\ ;\ v_{3,i}\}$; such as: $\begin{cases} v_{2,r} = e^{2k\pi v_{3,i}} \\ v_{3,i} = e^{\frac{\pi}{2} v_{3,i}} \end{cases}$

Its associated multiplication table is:

$\times$	$v_{2,r}$	$v_{3,i}$
$v_{2,r}$	$v_{2,r}$	$v_{3,i}$
$v_{3,i}$	$v_{3,i}$	$-v_{2,r}$

Table 2: multiplication table of base $\{v_{2,r}\ ;\ v_{3,i}\}$

The direct similarity matrix of a hyper complex number $\mathrm{h} = x_{2,r}v_{2,r} + y_{3,i}v_{3,i}$ of dimension 2 in the basis $\{v_{2,r}\ ;\ v_{3,i}\}$ is:

$$\mathcal{M}_2(h) = \begin{pmatrix} x_{2,r} & -y_{3,i} \\ y_{3,i} & x_{2,r} \end{pmatrix}$$

- **Euler basis $\{v_{3,r}\ ;\ v_{1,i}\}$**

The hyper-complex basis $\{v_{3,r}\ ;\ v_{1,i}\}$ is defined as follows:

- $e^{\theta v_{1,i}} = v_{2,r}\cos\theta + v_{1,i}\sin\theta$
- $e^{2k\pi v_{1,i}} = v_{3,r}$ we take $k = 0$
- $e^{\frac{\pi}{2} v_{1,i}} = v_{1,i}$
- $e^{\pi v_{1,i}} = -v_{1,r}$

The Euler identity in the basis $\{v_{3,r}\ ;\ v_{1,i}\}$ becomes $e^{\pi v_{1,i}} + v_{3,r} = 0$

Likewise we obtain its Euler basis $\{v_{3,r}\ ;\ v_{1,i}\}$; such as: $\begin{cases} v_{3,r} = e^{2k\pi v_{1,i}} \\ v_{1,i} = e^{\frac{\pi}{2} v_{1,i}} \end{cases}$

Its associated multiplication table is:

$\times$	$v_{3,r}$	$v_{1,i}$
$v_{3,r}$	$v_{3,r}$	$v_{1,i}$
$v_{1,i}$	$v_{1,i}$	$v_{3,r}$

Table 3: multiplication table of base $\{v_{3,r}\ ;\ v_{1,i}\}$

The direct similarity matrix of a hyper-complex number $\mathrm{h} = x_{3,r}v_{3,r} + y_{1,i}v_{1,i}$ of dimension 2 in the basis $\{v_{3,r}\ ;\ v_{1,i}\}$ is:

$$\mathcal{M}_2(h) = \begin{pmatrix} x_{3,r} & -y_{1,i} \\ y_{1,i} & x_{3,r} \end{pmatrix}$$

To conclude this part we will note:

$\mathbb{S}_{x_1,y_2}$ The algebra associated with the base $\{v_{1,r}\ ;\ v_{2,i}\}$ with coefficient in $\mathbb{R}$ to designate the set $\mathbb{S}_{(1;1)}$ of the affixes h_{x_1,y_2} of the points of the complex hyper-plane $(O\ ;\ v_{1,r}\ ;\ v_{2,i})$.

$\mathbb{S}_{x_2,y_3}$ The algebra associated with the base $\{v_{2,r}\ ;\ v_{3,i}\}$ with coefficient in $\mathbb{R}$ to designate the set $\mathbb{S}_{(1;1)}$ of the affixes h_{x_2,y_3} of the points of the complex hyper-plane $(O\ ; v_{2,r}\ ;\ v_{3,i})$.

$\mathbb{S}_{x_3,y_1}$ The algebra associated with the base $\{v_{3,r}\ ;\ v_{1,i}\}$ with coefficient in $\mathbb{R}$ to designate the set $\mathbb{S}_{(1;1)}$of the affixes h_{x_3,y_1} of the points of the complex hyper-plane $(O\ ; v_{3,r}\ ;\ v_{1,i})$.

3 ALGEBRAIC SPACE $\mathbb{S}_{(3;3)}$

3.1 Hyper-space $E_{(3;3)}$

We can define a complex hyper-space $E_{(3;3)}$ of dimension 6 (3 real dimensions and 3 imaginary dimensions), as being a geometric space constituted respectively by the orthogonal superposition of the complex hyper-planes $(O\ ;\ v_{1,r}\ ;\ v_{2,i})$; $(O\ ; v_{2,r}\ ;\ v_{3,i})$ and $(O\ ; v_{3,r}\ ;\ v_{1,i})$ in their direct directions of rotation and in the order of the respective "Euler bases" $\{v_{1,r}\ ;\ v_{2,i}\}$;

$\{v_{2,r}\ ;\ v_{3,i}\}$ and $\{v_{3,r}\ ;\ v_{1,i}\}$.

Then we can associate at any point in the complex hyperspace $E_{(3;3)}$ with 3 real dimensions and 3 imaginary dimensions $\{(v_{1,r}\ ; v_{2,r}\ ; v_{3,r}); (v_{1,i}\ ;\ v_{2,i}\ ;\ v_{3,i})\}$; a hyper-complex number h. This number being the sum of a number h_{x_1,y_2} affix of a point of the complex hyper-plane $(O\ ; v_{1,r}\ ;\ v_{2,i})$, of a number h_{x_2,y_3} affix of a second point of the hyper-complex plane $(O\ ; v_{2,r}\ ;\ v_{3,i})$ and a number h_{x_3,y_1} affix of a third point of the complex hyper-plane $(O\ ; v_{3,r}\ ;\ v_{1,i})$; thus defined in the direct order of their directions of rotation such as:

$h_{x_1,y_2} + h_{x_2,y_3} + h_{x_3,y_1}$ where $h_{x_1,y_2} \in \mathbb{S}_{x_1,y_2}$; $h_{x_2,y_3} \in \mathbb{S}_{x_2,y_3}$ and $h_{x_3,y_1} \in \mathbb{S}_{x_3,y_1}$.

We then have: $h = (x_{1,r}v_{1,r} + y_{2,i}v_{2,i}) + (x_{2,r}v_{2,r} + y_{3,i}v_{3,i}) + (x_{3,r}v_{3,r} + y_{1,i}v_{1,i})$

$h = x_{1,r}v_{1,r} + y_{2,i}v_{2,i} + x_{2,r}v_{2,r} + y_{3,i}v_{3,i} + x_{3,r}v_{3,r} + y_{1,i}v_{1,i}$

$h = x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r} + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

So the writing of the number h includes two parts:

A real part $x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r}$

An imaginary part $y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

So if we put $q_r = x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r}$ and $q_i = y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

We have: $h = q_r + q_i$

- q_r is a linear combination with real base of dimension 3.
- q_i is a linear combination with an imaginary base of dimension 3.

We will thus say that the "superimposed hyper-complex" numbers h of dimension 6 are obtained geometrically by orthogonal superposition. So by definition the set $\mathbb{S}_{(3;3)}$ is the set of "superimposed hyper-complex" numbers h of dimension 6.

We can therefore retain that the complex space $E_{(3;3)}$ is geometrically the superposition of two 3 dimensional geometric spaces: a real space of dimension 3 with real bases $\{v_{1,r}\ ; v_{2,r}\ ; v_{3,r}\}$ and an imaginary space of dimension 3 with imaginary bases $\{v_{1,i}\ ;\ v_{2,i}\ ;\ v_{3,i}\}$.

We note $E_{(3;3)} = \{(v_{1,r}\ ; v_{2,r}\ ; v_{3,r}); (v_{1,i}\ ;\ v_{2,i}\ ;\ v_{3,i})\}$.

We can clearly see that with this geometric and orthogonal superposition we obtain a well-defined writing of "hyper-complex superimposed" numbers. However, if we posit the equality $x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r} = x$ we obtain the Hamilton quaternions:

$q = x + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

Which can be interpreted by the fact that: the real space of quaternions is a space where real dimensions are intertwined.

This is why the set of "superimposed hyper-complex" numbers should be denoted $\mathbb{H}_S$ in honor of Hamilton who discovered them almost two centuries ago. It is true that he, himself was aware that writing hyper-complex works was only possible in this form. As if, in his subconscious, he knew that his numbers described states of superposition. What is certain is that if at that time, we knew the states of quantum superposition; he would have such an interpretation of his discovery.

To better visualize the construction of all hyper-complex numbers by geometric superposition; let us note in a circular manner:

For real bases $\begin{cases} v_{1,r} = 1_x \\ v_{2,r} = 1_y \\ v_{3,r} = 1_z \end{cases}$ and for imaginary bases $\begin{cases} v_{1,i} = i_x \\ v_{2,i} = i_y \\ v_{3,i} = i_z \end{cases}$ with $\begin{cases} {i_y}^2 = -1_x \\ {i_z}^2 = -1_y \\ {i_x}^2 = -1_z \end{cases}$

Under these conditions, we will see that we can obtain associative, commutative, unitary and bilinear algebra. However, for the rest of our study, let us keep the notations $\{v_{1,r}\ ; v_{2,r}\ ; v_{3,r}\}$ for the real bases and $\{v_{1,i}\ ;\ v_{2,i}\ ;\ v_{3,i}\}$ for the imaginary bases.

3.2 Equality of two "superimposed hyper-complex" numbers.

Two "superimposed hyper-complex" numbers are equal if and only if they have the same real bet and the same imaginary part.

Considering $h = q_r + q_i$ and $h' = {q_r}' + {q_i}'$ then: $h = h' \iff (q_r = {q_r}'$ et $q_i = {q_i}')$

$$h = h' \quad \iff \quad \begin{cases} x_{1,r} = {x_{1,r}}' \\ x_{2,r} = {x_{2,r}}' \\ x_{3,r} = {x_{3,r}}' \end{cases} \text{and} \quad \begin{cases} y_{1,i} = {y_{1,i}}' \\ y_{2,i} = {y_{2,i}}' \\ y_{3,i} = {y_{3,i}}' \end{cases}$$

So we can give the following definitions:

- Two "superimposed hyper-complex" numbers are said to be "equal-real" if they have the same real part and different imaginary parts.

Considering $h = q_r + q_i$ and $h' = {q_r}' + {q_i}'$ then:

h and h' are said to be "equal-real" if and only if $q_r = {q_r}'$ and $q_i \neq {q_i}'$

- Two "superimposed hyper-complex" numbers are said to be "equal-imaginary" if they have the same imaginary part and different real parts.

Considering $h = q_r + q_i$ and $h' = {q_r}' + {q_i}'$ then:

h and h' are said to be "equal-imaginary" if and only if $q_r \neq {q_r}'$ and $q_i = {q_i}'$.

- Two “superimposed hyper-complex” numbers are said to be “distinct” if their real parts are different and if their imaginary parts are also different.

Considering $h = q_r + q_i$ and $h' = q_r' + q_i'$ then:
h and h' are said to be “distinct” if and only if $q_r \neq q_r'$ and $q_i \neq q_i'$.

3.3 Operation in $\mathbb{S}(3\,;3)$

By the following steps we will also show that $\mathbb{S}_{(3;3)}$ is an associative, commutative, bilinear and unitary algebra on the set of real numbers $\mathbb{R}$.

3.3.1 Addition (internal operation $+$)

Considering h_1 and h_2 two “superimposed hyper-complex” numbers of dimension 6 such that:
$h = q_r + q_i$ and $h' = q_r' + q_i'$
The addition of hyper complex numbers placed in superposition of dimension 6 is then defined by:
$h + h' = (q_r + q_i) + (q_r' + q_i')$
The addition of real and imaginary numbers being commutative and associative we have:
$h + h' = (q_r + q_r') + (q_i + q_i')$ where
$q_r + q_r' = (x_{1,r} + x_{1,r}')v_{1,r} + (x_{2,r} + x_{2,r}')v_{2,r} + (x_{3,r} + x_{3r}')v_{3,r}$
$q_i + q_i' = (y_{1,i} + y_{1,i}')v_{1,i} + (y_{2,i} + y_{2,i}')v_{2,i} + (y_{3,i} + y_{3,i}')v_{3,i}$
Thus, the rules of addition in $\mathbb{S}_{(3;3)}$ are the same as those in $\mathbb{S}_{(1;1)}$. Therefore the addition in $\mathbb{S}_{(3;3)}$ is a commutative, associative operation and admits $0_{\mathbb{S}}$ as a neutral element.

3.3.2 Multiplication (internal operation $\times$)

Considering h and h' two “superimposed hyper-complex” numbers of dimension 6 such that:
$h = q_r + q_h$ and $h' = q_r' + q_h'$
The multiplication of “superimposed hyper-complex” numbers of dimension 6 is then defined by:
$h \times h' = (q_r + q_h) \times (q_r' + q_h')$
$h \times h' = q_r \times q_r' + q_r \times q_h' + q_h \times q_r' + q_h \times q_h'$
Where:
$q_r \times q_r' = (x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r}) \times (x_{1,r}'v_{1,r} + x_{2,r}'v_{2,r} + x_{3,r}'v_{3,r})$
$q_r \times q_i' = (x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r}) \times (y_{1,i}'v_{1,i} + y_{2,i}'v_{2,i} + y_{3,i}'v_{3,i})$
$q_i \times q_r' = (y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}) \times (x_{1,r}'v_{1,r} + x_{2,r}'v_{2,r} + x_{3,r}'v_{3,r})$
$q_i \times q_i' = (y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}) \times (y_{1,i}'v_{1,i} + y_{2,i}'v_{2,i} + y_{3,i}'v_{3,i})$
We see that writing the product of the “superimposed hyper-complex” numbers h and h' is too long and requires a multiplication table for the bases.

3.3.3 Hyper-complex bases Multiplication rules

To establish the multiplication table of the bases of the set $\mathbb{S}_{(3;3)}$; we will use the exponential writing of “Euler bases”.

Base $\{v_{1,r}\,;\,v_{2,i}\}$: $\begin{cases} e^{2k\pi v_{2,i}} = v_{1,r} \\ e^{\frac{\pi}{2}v_{2,i}} = v_{2,i} \end{cases}$; $k = 0$ and with a rotation $\mathcal{R}_{\left(O\,;-\frac{\pi}{2}\right)}$: $v_{1,i} \xrightarrow{-\frac{\pi}{2}} v_{2,r}$

Base $\{v_{2,r}\,;\,v_{3,i}\}$: $\begin{cases} e^{2k\pi v_{3,i}} = v_{2,r} \\ e^{\frac{\pi}{2}v_{3,i}} = v_{3,i} \end{cases}$; $k = 0$ and with a rotation $\mathcal{R}_{\left(O\,;-\frac{\pi}{2}\right)}$: $v_{3,i} \xrightarrow{-\frac{\pi}{2}} v_{2,r}$

Base $\{v_{2,r}\,;\,v_{3,i}\}$: $\begin{cases} e^{2k\pi v_{3,i}} = v_{2,r} \\ e^{\frac{\pi}{2}v_{3,i}} = v_{3,i} \end{cases}$; $k = 0$ and with a rotation $\mathcal{R}_{\left(O\,;-\frac{\pi}{2}\right)}$: $v_{1,i} \xrightarrow{-\frac{\pi}{2}} v_{3,r}$

So if we know the axis of the imaginary base $e^{\frac{\pi}{2}v_{n,i}} = v_{n,i}$, we will use the rotation $\mathcal{R}_{\left(O\,;-\frac{\pi}{2}\right)}$ of center O of angle $-\frac{\pi}{2}$ to determine the axis of the real base corresponding $e^{2k\pi v_{n,i}}$ in the plane of rotation and a transformation T. The transformation T is defined by:
$T\left(\sum v_{n,i}\right) = \sum v_{n,r}$
Which gives $e^{2k\pi v_{n,i}} = T \circ \mathcal{R}_{\left(O\,;-\frac{\pi}{2}\right)}(v_{n,i})$

The difficulty here consists of first determining the plane of rotation then the direction of rotation. Let us not forget, that we are no longer in 3-dimensional visual space but in 6-dimensional hyperspace. However, for each case; I will propose a plan and a direction of rotation, to then give the orientation of the real bases corresponding to each imaginary base. On the other hand, the algebraic writing of Hamilton's quaternions is a perfect mathematical tool; which describes an intertwined real space, where the real bases can be everywhere. Thus, determining the orientation of real bases will no longer be a necessity in an interlaced real space. What we need to know is that; any direction perpendicular to the imaginary axis, is a possible direction for the corresponding real axis. However, the exponential writing of hyper-complex numbers allows us, to do any type of calculation without taking in account geometric aspect.

3.3.4 Calculation of type $(v_{n-1,r})^2$

$v_{n-1,r}$ is the real unit carried by the axis perpendicular to the axis of the imaginary unit $v_{n,i}$

We have $\left(v_{n-1,r}\right)^2 = v_{n-1,r} \times v_{n-1,r} = e^{2k\pi v_{n,i}} \times e^{2k\pi v_{n,i}} = e^{4k\pi v_{n,i}}$

If we take $k = 0$ then $e^{4k\pi v_{n,i}} = e^{2k\pi v_{n,i}} = e^{0 v_{n,i}} = v_{n-1,r}$

Therefore $\left(v_{n-1,r}\right)^2 = v_{n-1,r}$

3.3.5 Calculation type $(v_{n,i})^2$

We have $\left(v_{n,i}\right)^2 = v_{n,i} \times v_{n,i} = e^{\frac{\pi}{2}v_{n,i}} \times e^{\frac{\pi}{2}v_{n,i}} = e^{\pi v_{n,i}} = -v_{n-1,r}$

Calculation type $v_{n-1,r} \times v_{n,i} = v_{n,i} \times v_{n-1,r}$

$v_{n-1,r} \times v_{n,i} = e^{2k\pi v_{n,i}} \times e^{\frac{\pi}{2}v_{n,i}} = e^{\left(2k\pi v_{n,i} + \frac{\pi}{2}v_{n,i}\right)}$

If we take $k = 0$ then $e^{\left(0 v_{n,i} + \frac{\pi}{2}v_{n,i}\right)} = e^{\frac{\pi}{2}v_{n,i}} = v_{n,i}$

Therefore $v_{n-1,r} \times v_{n,i} = v_{n,i}$

So we can retain that multiplying a real base by an imaginary base always gives us the imaginary base.

3.3.6 Calculations of types $s_{n,r} = v_{n,r} \times v_{n-1,r}$ and $s_{n,i} = v_{n,i} \times v_{n-1,i}$

We start by calculating $s_{n,i} = v_{n,i} \times v_{n-1,i}$ then we deduce $s_{n,r} = v_{n,r} \times v_{n-1,r}$ geometrically by the rotation $\mathcal{R}_{\left(O\,;\,-\frac{\pi}{2}\right)}$ with center O and angle $-\frac{\pi}{2}$ and by the transformation T.

3.3.7 Euler Base $\{s_{1,r}\ ;\ s_{2,i}\}$ of module $\rho_2 = \sqrt{2}$

$s_{2,i} = v_{1,i} \times v_{2,i} = v_{2,i} \times v_{1,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}$

$s_{1,r} = v_{1,r} \times v_{3,r} = v_{3,r} \times v_{1,r} = e^{2k\pi v_{2,i}} \times e^{2k\pi v_{1,i}} = e^{2k\pi\left(v_{1,i}+v_{2,i}\right)}$.

However we have: $k = 0$ so $s_{1,r} = e^{0\left(v_{1,i}+v_{2,i}\right)}$

We see that $\left(s_{2,i}\right)^2 = \left(e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}\right)^2 = e^{\pi\left(v_{1,i}+v_{2,i}\right)} = -s_{1,r}$

We see the imaginary number $s_{2,i}$ appear such that: $\left(s_{2,i}\right)^2 = -s_{1,r}$

So we can write using Euler's formula:

- $e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = e^{\frac{\pi}{2}s_{2,i}} = \left(s_{1,r}\cos\frac{\pi}{2} + s_{2,i}\sin\frac{\pi}{2}\right) = s_{2,i}$
- $e^{2k\pi(v_{1,i}+v_{2,i})} = e^{2k\pi s_{2,i}} = \left(s_{1,r}\cos 2k\pi + s_{2,i}\sin 2k\pi\right) = s_{1,r}$

We must then define the existence of a new real and imaginary base $\{s_{1,r}\ ;\ s_{2,i}\}$ of module:

- $\rho_{s_{2,i}} = \rho_2 = \left|s_{2,i}\right| = \left|e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}\right| = \left|v_{1,i} + v_{2,i}\right| = \sqrt{2}$
- $\rho_{s_{1,r}} = \left|s_{1,r}\right| = \left|e^{0(v_{1,i}+v_{2,i})}\right| = \left|\left(e^{v_{1,i}+v_{2,i}}\right)^0\right| = |1| = 1$

We can determine; geometrically the orientation of the imaginary number by:

$$s_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = v_{1,i} + v_{2,i}$$

Finally to determine the orientation of the real number $s_{1,r}$ of the base $\{s_{1,r}\ ;\ s_{2,i}\}$, we must consider the rotation $\mathcal{R}_{\left(O\,;\,-\frac{\pi}{2}\right)}$ in the rotation plane $\left(O\ ;\ v_1\ ;\ v_2\right)$ and the transformation

$$T\left(\sum v_{n,i}\right) = \sum v_{n,r}$$

We find geometrically:

$$T \circ \mathcal{R}_{\left(O\,;\,-\frac{\pi}{2}\right)}\left(v_{1,i} + v_{2,i}\right) = v_{1,r} - v_{2,r}$$

Thus geometrically, the orientation of the real number $s_{1,r}$ is the same as the number $v_{1,r} - v_{2,r}$.

We obtain the base $\{s_{1,r}\ ;\ s_{2,i}\}$ with imaginary module $\rho_2 = \sqrt{2}$ and real module 1 in the rotation plane $(O\ ;\ v_1\ ;\ v_2)$

Base $\{s_{1,r}\ ;\ s_{2,i}\}$: $\begin{cases} s_{1,r} = e^{2k\pi(v_{1,i}+v_{2,i})} \\ s_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \end{cases}$ with $k = 0$

We can say that, the number $s_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}$ describes a state of superposition of imaginary numbers $v_{1,i}$ and $v_{2,i}$. Which means that by deduction that: $s_{1,r} = e^{2k\pi(v_{1,i}+v_{2,i})}$ describes a state of superposition of real numbers $v_{1,r}$ and $v_{2,r}$.

3.3.8 Euler Base $\{s_{2,r}\ ;\ s_{3,i}\}$ and $\{s_{3,r}\ ;\ s_{1,i}\}$ of module $\rho_2=\sqrt{2}$

In the same way, we can deduce that:

$s_{3,i} = v_{2,i} \times v_{3,i} = v_{3,i} \times v_{2,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}v_{3,i}} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = v_{2,i} + v_{3,i}$

$s_{2,r} = v_{1,r} \times v_{2,r} = v_{2,r} \times v_{1,r} = e^{2k\pi v_{2,i}} \times e^{2k\pi v_{3,i}} = e^{2k\pi(v_{2,i}+v_{3,i})}$

We obtain a new base $\{s_{2,r}\ ;\ s_{3,i}\}$ with imaginary module $\rho_2 = \sqrt{2}$ and real module 1 in the rotation plane $(O\ ;\ v_2\ ;\ v_3)$.

Base $\{s_{2,r}\ ;\ s_{3,i}\}$: $\begin{cases} s_{2,r} = e^{2k\pi(v_{2,i}+v_{3,i})} \\ s_{3,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} \end{cases}$ with $k = 0$

Likewise we have:

$s_{1,i} = v_{1,i} \times v_{3,i} = v_{3,i} \times v_{1,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{3,i}} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} = v_{1,i} + v_{3,i}$

$s_{3,r} = v_{2,r} \times v_{3,r} = v_{3,r} \times v_{2,r} = e^{2k\pi v_{3,i}} \times e^{2k\pi v_{1,i}} = e^{2k\pi(v_{3,i}+v_{1,i})}$

We obtain a new base $\{s_{3,r}\ ;\ s_{1,i}\}$ with imaginary module $\rho_2 = \sqrt{2}$ and real module 1 in the rotation plane $(O\ ;\ v_3\ ;\ v_1)$.

Base $\{s_{3,r}\ ;\ s_{1,i}\}$: $\begin{cases} s_{3,r} = e^{2k\pi(v_{3,i}+v_{1,i})} \\ s_{1,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \end{cases}$ with $k = 0$

3.3.9 Set $\mathbb{S}_{(3;3)}$ Multiplication table

We can associate with multiplication (the × operation) of the $\mathbb{S}_{(3;3)}$ the following table:

$\times$	$v_{1,r}$	$v_{2,r}$	$v_{3,r}$	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$
$v_{1,r}$	$v_{1,r}$	$s_{2,r}$	$s_{1,r}$	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$
$v_{2,r}$	$s_{2,r}$	$v_{2,r}$	$s_{3,r}$	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$
$v_{3,r}$	$s_{1,r}$	$s_{3,r}$	$v_{3,r}$	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$
$v_{1,i}$	$v_{1,i}$	$v_{1,i}$	$v_{1,i}$	$-v_{3,r}$	$s_{2,i}$	$s_{1,i}$
$v_{2,i}$	$v_{2,i}$	$v_{2,i}$	$v_{2,i}$	$s_{2,i}$	$-v_{1,r}$	$s_{3,i}$
$v_{3,i}$	$v_{3,i}$	$v_{3,i}$	$v_{3,i}$	$s_{1,i}$	$s_{3,i}$	$-v_{2,r}$

Table 4: multiplication table of bases of the set $\mathbb{S}_{(3;3)}$

3.3.10 Euler base $\{u_{1,r}\ ;\ u_{2,i}\}$ of module $\rho_3=\sqrt{3}$

Calculation type: $u_{2,i} = (v_{n-1,i} \times v_{n,i}) \times v_{n+1,i} = v_{n-1,i} \times (v_{n,i} \times v_{n+1,i})$

And $u_{2,r} = (v_{n-1,r} \times v_{n,r} \times) v_{n+1,r} = v_{n-1,r} \times (v_{n,r} \times v_{n+1,r})$

We also obtain a new base $\{u_{1,r}\ ;\ u_{2,i}\}$ of module $\rho_3 = \sqrt{3}$ in the rotation plane $(O\ ;\ s_{2,i}\ ;\ v_3)$.

We have:

$u_{2,i} = (v_{1,i} \times v_{2,i}) \times v_{3,i} = v_{1,i} \times (v_{2,i} \times v_{3,i}) = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}v_{3,i}}$

$u_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}$

Likewise, we have:

$u_{1,r} = (v_{1,r} \times v_{2,r}) \times v_{3,r} = v_{1,r} \times (v_{2,r} \times v_{3,r}) = e^{2k\pi v_{2,i}} \times e^{2k\pi v_{3,i}} \times e^{2k\pi v_{1,i}}$

$u_{1,r} = e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}$

However we have: k=0 so $s_{1,r} = e^{0(v_{1,i}+v_{2,i}+v_{3,r})}$

We see that: $\left(u_{2,i}\right)^2 = \left(e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}\right)^2 = e^{\pi(v_{1,i}+v_{2,i}+v_{3,i})} = -u_{1,r}$

We see the imaginary number $u_{2,i}$ appear again such that: $\left(u_{2,i}\right)^2 = -u_{1,r}$
So we can write using Euler's formula:

- $e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}u_{2,i}} = \left(u_{1,r}\cos\frac{\pi}{2} + u_{2,i}\sin\frac{\pi}{2}\right) = u_{2,i}$
- $e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})} = e^{2k\pi u_{2,i}} = \left(u_{1,r}\cos 2k\pi + u_{2,i}\sin 2k\pi\right) = u_{1,r}$

We must then define the existence of a new real and imaginary base $\{u_{1,r}\ ;\ u_{2,i}\}$ of module:

- $\rho_{u_{2,i}} = \rho_3 = |u_{2,i}| = \left|e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}\right| = |v_{1,i}+v_{2,i}+v_{3,i}| = \sqrt{3};$
- $\rho_{u_{2,r}} = \rho_3 = |u_{2,r}| = \left|e^{0(v_{1,i}+v_{2,i}+v_{3,r})}\right| = \left|\left(e^{(v_{1,i}+v_{2,i}+v_{3,r})}\right)^0\right| = |1| = 1$
- We can determine; geometrically the orientation of the imaginary number by:

$$u_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = v_{1,i}+v_{2,i}+v_{3,i}$$

Finally to determine the orientation of the real number $u_{1,r}$ of the base $\{u_{1,r}\ ;\ u_{2,i}\}$, we must consider the rotation $\mathcal{R}_{\left(O\,;\,-\frac{\pi}{2}\right)}$ in the rotation plane$\left(O\ ;\ s_{2,i}\ ;\ v_3\right)$ and the transformation:

$$T\left(\sum v_{n,i}\right) = \sum v_{n,r}$$

We find geometrically:

$$T \circ \mathcal{R}_{\left(O\,;\,-\frac{\pi}{2}\right)}\left(v_{1,i}+v_{2,i}+v_{3,i}\right) = v_{1,r}+v_{2,r}-v_{3,r}$$

Thus geometrically, the orientation of the real number $u_{1,r}$ of module 1 is the same as the number $v_{1,r}+v_{2,r}-v_{3,r}$.
We obtain the base $\{u_{1,r}\ ;\ u_{2,i}\}$ with imaginary module $\rho_3 = \sqrt{3}$ and real module 1 in the rotation plane $\left(O\ ;\ s_{2,i}\ ;\ v_3\right)$.

We obtain the base $\{u_{1,r}\ ;\ u_{2,i}\}$: $\begin{cases} u_{1,r} = e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})} \\ u_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} \end{cases}$ with $k = 0$

Their geometric interpretation is: in my opinion:

- $u_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}$: describes a state of superposition of imaginary numbers $v_{1,i}$; $v_{2,i}$ and $v_{3,i}$
- $u_{1,r} = e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}$: describes a state of superposition of real numbers $v_{1,r}$; $v_{2,r}$ and $v_{3,r}$

3.4 Priority rules

Now we need to define some priority rules for multiplication with the number –1. Since the works of Jean Robert Argand, of Jean Fréderic Français; we know that multiplying by -1 amounts to making a rotation of angle π in the same plane of rotation. What happens if we multiply two numbers located in two different planes of rotation?

We know that in the same plane we have: $e^{-\frac{\pi}{2}v_{1,i}} = -e^{\frac{\pi}{2}v_{1,i}}$ and $-e^{\frac{\pi}{2}v_{1,i}} = e^{-\frac{\pi}{2}v_{1,i}}$

Thus: $-e^{\frac{\pi}{2}v_{1,i}} \times -e^{\frac{\pi}{2}v_{1,i}} = \left(e^{\frac{\pi}{2}v_{1,i}}\right)^2 = e^{\pi v_{1,i}} = -e^{2k\pi v_{1,i}}$

Likewise: $-e^{\frac{\pi}{2}v_{1,i}} \times \left(-e^{\frac{\pi}{2}v_{1,i}}\right) = e^{-\frac{\pi}{2}v_{1,i}} \times e^{-\frac{\pi}{2}v_{1,i}} = e^{-\pi v_{1,i}} = e^{\pi v_{1,i}} = -e^{2k\pi v_{1,i}}$

In both cases, we find the same result.
However, if we take two numbers of different rotation planes we will have:

First: $-e^{\frac{\pi}{2}v_{1,i}} \times -e^{\frac{\pi}{2}v_{2,i}} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}$

Then secondly: $-e^{\frac{\pi}{2}v_{1,i}} \times -e^{\frac{\pi}{2}v_{2,i}} = e^{-\frac{\pi}{2}v_{1,i}} \times e^{-\frac{\pi}{2}v_{2,i}} = e^{-\frac{\pi}{2}(v_{1,i}+v_{2,i})} = -e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}$

We see that the two approaches give us two opposite results. Here we are faced with the same problem with the imaginary notation $\sqrt{-1}$; before Leonard Euler proposed the imaginary notation i to replace the number $\sqrt{-1}$. Hence, the need to define a priority rule for the exponential e. So keep the second approach as a rule. Not only is it the exponential e which gives the orientation of the imaginary axes but also it is this approach which verifies Euler's formula. Unless we define a new rule of signs linked to complex superposition; for non-coplanar bases which, as we know, describe superposition states. If we give priority to the exponential e; means that before multiplying we must raise the negative sign $(-)$ on the exponential e. After all that, all that remains is to multiply positive coefficients $(+)$. This is not surprising, because we know that: $-i = \frac{1}{i}$. Which shows that there is no difference between $-i$ and $\frac{1}{i}$. Hence, the need to define a priority rule for multiplying different imaginary bases.

Therefore, we must also know that; this priority given to the exponential e; has the merit of increasing the number of intermediate and higher bases.

3.4.1 Complementary secondary bases.

So, if we have to multiply:

$$e^{\frac{\pi}{2}v_{1,i}} \times \left(-e^{\frac{\pi}{2}v_{2,i}}\right) = e^{\frac{\pi}{2}v_{1,i}} \times e^{-\frac{\pi}{2}v_{2,i}} = e^{\frac{\pi}{2}(v_{1,i} - v_{2,i})}$$

We thus obtain a new imaginary number $e^{\frac{\pi}{2}(v_{1,i} - v_{2,i})}$; whose real base is $e^{2k\pi(v_{1,i} - v_{2,i})}$

We must also know that; the numbers $e^{\frac{\pi}{2}(v_{1,i} - v_{2,i})}$ and $e^{\frac{\pi}{2}(v_{2,i} - v_{1,i})}$ are opposite imaginaries. Because

$$e^{\frac{\pi}{2}(v_{2,i} - v_{1,i})} = e^{-\frac{\pi}{2}(v_{1,i} - v_{2,i})} = -e^{\frac{\pi}{2}(v_{1,i} - v_{2,i})}$$

Likewise we will have the imaginaries $e^{\frac{\pi}{2}(v_{2,i} - v_{3,i})}$ and $e^{\frac{\pi}{2}(v_{3,i} - v_{1,i})}$ of respective real bases $e^{2k\pi(v_{2,i} - v_{3,i})}$ and $e^{2k\pi(v_{3,i} - v_{1,i})}$.

We obtain three more new secondary bases; thanks to this priority rule.

- $s_{4,r} = e^{2k\pi(v_{1,i} - v_{2,i})}$ and $s_{5,i} = e^{\frac{\pi}{2}(v_{1,i} - v_{2,i})}$
- $s_{5,r} = e^{2k\pi(v_{2,i} - v_{3,i})}$ and $s_{6,i} = e^{\frac{\pi}{2}(v_{2,i} - v_{3,i})}$
- $s_{6,r} = e^{2k\pi(v_{3,i} - v_{1,i})}$ and $s_{4,i} = e^{\frac{\pi}{2}(v_{3,i} - v_{1,i})}$

What we can add is that; the rotation planes of these new bases are inclined in relation to the planes of the intermediate bases already proposed. A comparison would be necessary, with the planes of rotation of electrons around an atom, to have a precise idea of all the planes of rotation of hyper-complex bases.

3.4.2 The complementary upper bases.

If we have to multiply:

$$e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} \times \left(-e^{\frac{\pi}{2}v_{3,i}}\right) = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} \times e^{-\frac{\pi}{2}v_{3,i}} = e^{\frac{\pi}{2}(v_{1,i} + v_{2,i} - v_{3,i})}$$

We thus obtain a new imaginary number $e^{\frac{\pi}{2}(v_{1,i} + v_{2,i} - v_{3,i})}$ whose real base is $e^{2k\pi(v_{1,i} + v_{2,i} - v_{3,i})}$.

Likewise we will have the imaginaries $e^{\frac{\pi}{2}(v_{2,i} + v_{3,i} - v_{1,i})}$ and $e^{\frac{\pi}{2}(v_{3,i} + v_{1,i} - v_{2,i})}$ of respective real bases $e^{2k\pi(v_{2,i} + v_{3,i} - v_{1,i})}$ and $e^{2k\pi(v_{3,i} + v_{1,i} - v_{2,i})}$.

We obtain three more new upper bases.

- $u_{2,r} = e^{2k\pi(v_{1,i} + v_{2,i} - v_{3,i})}$ and $u_{3,i} = e^{\frac{\pi}{2}(v_{1,i} + v_{2,i} - v_{3,i})}$
- $u_{3,r} = e^{2k\pi(v_{2,i} + v_{3,i} - v_{1,i})}$ and $u_{4,i} = e^{\frac{\pi}{2}(v_{2,i} + v_{3,i} - v_{1,i})}$
- $u_{4,r} = e^{2k\pi(v_{3,i} + v_{1,i} - v_{2,i})}$ and $u_{1,i} = e^{\frac{\pi}{2}(v_{3,i} + v_{1,i} - v_{2,i})}$

So:

The opposite of the imaginary number $e^{\frac{\pi}{2}(v_{1,i} + v_{2,i} - v_{3,i})}$ is $e^{\frac{\pi}{2}(v_{3,i} - v_{1,i} - v_{2,i})}$

The opposite of the imaginary number $e^{\frac{\pi}{2}(v_{2,i} + v_{3,i} - v_{1,i})}$ is $e^{\frac{\pi}{2}(v_{1,i} - v_{2,i} - v_{3,i})}$

The opposite of the imaginary number $e^{\frac{\pi}{2}(v_{3,i} + v_{1,i} - v_{2,i})}$ is $e^{\frac{\pi}{2}(v_{2,i} - v_{3,i} - v_{1,i})}$

3.4.3 General expression of a "superimposed hyper-complex" number h

We can give a general algebraic formula for "superimposed hyper-complex" numbers; expressed with primary, intermediate and higher bases, in 6-dimensional hyperspace. This general formula is given by:

$$h = q_r + q_i \quad \text{where} \quad : \begin{cases} q_r = \sum_{n=1}^{3} x_{n,r} v_{n,r} + \sum_{n=1}^{6} a_{n,r} s_{n,r} + \sum_{n=1}^{4} c_{n,r} u_{n,r} \\ q_i = \sum_{n=1}^{3} y_{n,i} v_{n,i} + \sum_{n=1}^{6} b_{n,i} s_{n,i} + \sum_{n=1}^{4} d_{n,i} u_{n,i} \end{cases}$$

This general expression of h; whose writing includes the primary, intermediate and higher bases; allows you to perform algebraic calculations with "hyper-complex superimposed" numbers in 6 dimensions without taking into account their geometric aspect. What is certain is that; for dimensions $n \geq 4$ even locating the orientation and direction of the imaginary bases requires; extensive knowledge of the properties of associated hyperspaces; such as the concept of hyper-angle. We can thus say; that the three (03) primary complex bases hide six (06) intermediate complex bases and four (04) upper complex bases in the set $\mathbb{S}_{(3;3)}$.

3.5 Orbitals of set $\mathbb{S}_{(3;3)}$

We note that the set $\mathbb{S}_{(3;3)}$ has:

- Three (03) primary complex bases of module $\rho_1 = \sqrt{1} = 1 : \{v_{1,r} ; v_{2,i}\} ; \{v_{2,r} ; v_{3,i}\}$ and $\{v_{3,r} ; v_{1,i}\}$. The orbitals with primary complex bases are circular with radius $r = 1$.
- Six (06) intermediate imaginary bases of module $\rho_2 = \sqrt{2}$ and real module $\rho_r = 1$: $\{s_{1,r} ; s_{2,i}\} ; \{s_{2,r} ; s_{3,i}\} ; \{s_{3,r} ; s_{1,i}\} ; \{s_{4,r} ; s_{5,i}\} ; \{s_{5,r} ; s_{6,i}\}$ and $\{s_{6,r} ; s_{1,i}\}$. All are linear combinations of primary bases two by two. The orbitals with intermediate complex bases are elliptical with semi-major axis $a = \rho_2 = \sqrt{2}$ and semi-minor axis $b = 1$.
- Four (04) upper imaginary bases of module $\rho_3 = \sqrt{3}$ and real module $\rho_r = 1$: $\{u_{1,r} ; u_{2,i}\} ; \{u_{2,r} ; u_{3,i}\} ; \{u_{3,r} ; u_{4,i}\}$ and $\{u_{4,r} ; u_{1,i}\}$. All are linear combinations of the three (03) primary bases. The orbitals with intermediate complex bases are elliptical with semi-major axis $a = \rho_3 = \sqrt{3}$ and semi-minor axis $b = 1$.

We can say that the bases of the complex hyperspace $E_{(3\,;\,3)}$ are distributed over thirteen (13) levels. If we consider these levels as orbitals, we have:

- Three (03) circular orbitals of radius $\rho_1 = \sqrt{1} = 1$.
- Six (06) elliptical orbitals of radius with semi-major axis $a = \rho_2 = \sqrt{2}$ and semi-minor axis $b = 1$.
- Four (04) elliptical orbital with semi-major axis $a = \rho_3 = \sqrt{3}$ and semi-minor axis $b = 1$.

Which gives an analogy with the structure of the possible orbitals of electrons, which orbit the nucleus of an atom. What is certain; is that if the hyperspace orbitals $E_{(3\,;\,3)}$ are not sufficient to explain the distribution of electrons around the nucleus; there exist higher spaces $E_{(9\,;\,9)}$; $E_{(27\,;\,27)}$; …….. ; $E_{(n\,;\,n)}$ and $E_{(3n\,;\,3n)}$; which we will define in the following parts.

3.6 Applications “real product” $\otimes^r_e$ and “imaginary product” $\otimes^i_e$ to determine the product of hyper-complex bases.

To simplify the calculations, let us define the applications, which we will call Euler products; the products noted: $\otimes_e^r$ the real product and $\otimes_e^i$ the imaginary product.

3.6.1 The imaginary product $\otimes^i_e$ rules

Considering $V_i = e^{\frac{\pi}{2}(\alpha_1 v_{1,i} + \alpha_2 v_{2,i} + \alpha_3 v_{3,i})}$ and $V_i' = e^{\frac{\pi}{2}(\alpha_1{}' v_{1,i} + \alpha_2{}' v_{2,i} + \alpha_3{}' v_{3,i})}$

The imaginary product $\otimes_e^i$ of the two imaginary bases V_i and V_i' is:

$\otimes_e^i(V_i \,;\, V_i') = e^{\frac{\pi}{2}(\alpha_1 v_{1,i} + \alpha_2 v_{2,i} + \alpha_2 v_{2,i})} \times e^{\frac{\pi}{2}(\alpha_1{}' v_{1,i} + \alpha_2{}' v_{2,i} + \alpha_3{}' v_{3,i})}$ where $\alpha_k \in \{0 \,;\, 1\}$.

We must then define the following applications, which we will associate with the imaginary product $\otimes_e^i$:

Considering the sets $H = \{0 + 4p \,;\, 1 + 4p \,;\, 2 + 4p \,;\, 3 + 4p\}$ $(p \in \mathbb{Z})$; $I = \{-1 \,;\, 0 \,;\, 1\}$ and $J = \{-1 \,;\, +1\}$ and the applications; f and $S_{n,\pm}$ defined by:

- $f(H) = I$ such as: $\begin{cases} f(0 + 4p) = 0 \\ f(1 + 4p) = 1 \\ f(2 + 4p) = 0 \\ f(3 + 4p) = -1 \end{cases}$
- $S_{n,\pm} = \left(\{0 + 4p \,;\, 2 + 4p\}\right) = J$ such that: $S_{n,\pm} = \begin{cases} S_{0,\pm}(0 + 4p) = +1 \\ S_{2,\pm}(2 + 4p) = -1 \end{cases}$

Then the imaginary product application is defined by:

$\otimes_e^i(V_i \,;\, V_i') = e^{\frac{\pi}{2}(\alpha_1 v_{1,i} + \alpha_2 v_{2,i} + \alpha_2 v_{2,i})} \times e^{\frac{\pi}{2}(\alpha_1{}' v_{1,i} + \alpha_2{}' v_{2,i} + \alpha_3{}' v_{3,i})}$

$\otimes_e^i(V_i \,;\, V_i') = (S_{0,\pm} \times S_{2,\pm}) \times e^{\frac{\pi}{2}[f(\alpha_1 + \alpha_1{}') \times v_{1,i} + f(\alpha_2 + \alpha_2{}') \times v_{2,i} + f(\alpha_3 + \alpha_3{}') \times v_{3,i}]}$

And we have:

$\otimes_e^i(V_i \,;\, V_i') = (S_{0,\pm} \times S_{2,\pm}) \times e^{\frac{\pi}{2} \times [f(1+4p) \sum_{n=1}^{3} v_{n,i} + f(3+4p) \sum_{m=1}^{3} v_{m,i}]}$

If we note $\prod_i = S_{0,\pm} \times S_{2,\pm}$ we will have:

$\otimes_e^i(V_i \,;\, V_i') = \prod_i \times e^{\frac{\pi}{2} \times [f(1+4p) v_{n,i} + f(3+4p) v_{p,i}]}$

3.6.2 Real Product Rules

We geometrically deduce the corresponding real product by the rotation $R_{\left(0\,;\,-\frac{\pi}{2}\right)}$ of the imaginary product and by the transformation T.

We have: $\otimes_e^r(V_r \,;\, V_r{}') = e^{2k\pi(\alpha_1 v_{1,i} + \alpha_2 v_{2,i} + \alpha_3 v_{3,i})} \times e^{2k\pi(\alpha_1{}' v_{1,i} + \alpha_2{}' v_{2,i} + \alpha_3{}' v_{3,i})}$

$\otimes_e^r(V_r\,;V_r{}') = T \circ R_{\left(O\,;-\frac{\pi}{2}\right)}\left(\Pi_i \times \left[e^{\frac{\pi}{2}(\alpha_1 v_{1,i}+\alpha_2 v_{2,i}+\alpha_3 v_{3,i})}\right] \times T \circ R_{\left(O\,;-\frac{\pi}{2}\right)}\left[e^{\frac{\pi}{2}(\alpha_1{}' v_{1,i}+\alpha_2{}' v_{2,i}+\alpha_3{}' v_{3,i})}\right]\right)$

$\otimes_e^r(V_r\,;\,V_r{}') = T \circ R_{\left(O\,;-\frac{\pi}{2}\right)}\left(\Pi_i \times \left[e^{\frac{\pi}{2}(\alpha_1 v_{1,i}+\alpha_2 v_{2,i}+\alpha_3 v_{3,i})} \times e^{\frac{\pi}{2}(\alpha_1{}' v_{1,i}+\alpha_2{}' v_{2,i}+\alpha_3{}' v_{3,i})}\right]\right)$

So we have now: $\otimes_e^r(V_r\,;\,V_r{}') = T \circ R_{\left(O\,;-\frac{\pi}{2}\right)}\left[\Pi_i \times \otimes_e^r(V_i\,;\,V_i{}')\right]$

With $R_{\left(O\,;-\frac{\pi}{2}\right)}$ the rotation with center O and angle $-\frac{\pi}{2}$ in the plane of rotation and T the transformation defined by $T\left(\sum v_{n,i}\right) = \sum v_{n,r}$.

3.6.3 Examples:

- **Example1:**

Let us calculate $s_{2,i} \times s_{3,i}$:

We can put $s_{2,i} \times s_{3,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})}$

$$s_{2,i} \times s_{3,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}v_{3,i}}$$

$$s_{2,i} \times s_{3,i} = e^{\frac{\pi}{2}v_{1,i}} \times \left(e^{\frac{\pi}{2}v_{2,i}}\right)^2 \times e^{\frac{\pi}{2}v_{3,i}}$$

$$s_{2,i} \times s_{3,i} = -e^{\frac{\pi}{2}(v_{1,i}+v_{3,i})}$$

$$s_{2,i} \times s_{3,i} = -s_{1,i}$$

If we use the imaginary product, we have:

$$s_{2,i} \times s_{3,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})}$$

$$s_{2,i} \times s_{3,i} = \otimes_e^i\left[e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})}\right]$$

$$s_{2,i} \times s_{3,i} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}+v_{3,i})}$$

$$s_{2,i} \times s_{3,i} = e^{-\frac{\pi}{2}(v_{1,i}+v_{3,i})}$$

$$s_{2,i} \times s_{3,i} = -s_{1,i}$$

In both cases, we find the same result.

It should be noted that; the purpose of the imaginary product is to bring out from the exponential, the real ones which are formed inside by cycle. Its interest is; to allow us to do calculations without taking into account the multiplication table.

Noticed:

If $f(2+4p)$ appears twice (or more) give priority to the corresponding intermediate imaginary base (or upper base).

For example if we have:

$$u_{2,i} \times s_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}$$

$$u_{2,i} \times s_{2,i} = e^{\frac{\pi}{2}(2v_{1,i}+2v_{2,i}+v_{3,i})}$$

$$u_{2,i} \times s_{2,i} = e^{\frac{\pi}{2}[2(v_{1,i}+v_{2,i})+v_{3,i}]}$$

$$u_{2,i} \times s_{2,i} = e^{\frac{\pi}{2}[2(v_{1,i}+v_{2,i})+v_{3,i}]}$$

$$u_{2,i} \times s_{2,i} = e^{\frac{\pi}{2}[2s_{2,i}+v_{3,i}]}$$

$$u_{2,i} \times s_{2,i} = e^{-\frac{\pi}{2}v_{3,i}}$$

$$u_{2,i} \times s_{2,i} = -v_{3,i}$$

We must notice that: if we give priority to the intermediate and upper imaginary bases, we lose the associativity of the imaginary product. However, as I specified in the paragraph (4.2.10 p21), if we define a multiplication sign rules linked to complex superposition, the property of associativity will be preserved.

- **Example 2:**

Let us calculate $s_{3,r} \times s_{2,r}$

$$s_{3,r} \times s_{2,r} = \otimes_e^r\left(e^{2k\pi(v_{3,i}+v_{1,i})}\,;\,e^{2k\pi(v_{2,i}+v_{3,i})}\right)$$

$$s_{3,r} \times s_{2,r} = T \circ R_{\left(O\,;-\frac{\pi}{2}\right)}\left[\Pi_i \times \otimes_e^i\left(e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})}\,;\,e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})}\right)\right]$$

$$s_{3,r} \times s_{2,r} = T \circ R_{\left(O\,;-\frac{\pi}{2}\right)}\left[\Pi_i \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+2v_{3,i})}\right]$$

$$s_{3,r} \times s_{2,r} = T \circ R_{\left(O\,;\,-\frac{\pi}{2}\right)}\left[(-) \times (-)e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}\right]$$
$$s_{3,r} \times s_{2,r} = T \circ R_{\left(O\,;\,-\frac{\pi}{2}\right)}\left[e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}\right]$$
$$s_{3,r} \times s_{2,r} = e^{2k\pi(v_{1,i}+v_{2,i})}$$
$$s_{3,r} \times s_{2,r} = s_{1,r}$$

We must take into account, that all the primary, intermediate and upper real bases have the same module. All their module are equal to 1. Which means that the purpose of the real product is not only to make a calculation but to determine their geometric orientations.

3.7 Hyper-plane $P_{(3;\,3)}$ associated with the hyper-space $E_{(3;\,3)}$ of dimension 6.

We know that a hyper-complex number of dimension 6 is written in the form:

$h = (x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r}) + (y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i})$

If $q_r = x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r}$ and $q_i = y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

Then $h = q_r + q_i$ where q_r is a real number of dimension 3 and q_i an imaginary number of dimension 3. We have the equality:

$u_{1,r} = v_{1,r} \times v_{2,r} \times v_{3,r} = e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})} = v_{1,r} + v_{2,r} - v_{3,r}$ (with $k = 0$).

The number $u_{1,r}$ designates the real base of module $\rho = \sqrt{3}$.

So: $q_r = q_r \times {u_{1,r}}^{-1} \times u_{1,r}$

$$q_r = (x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r}) \times {u_{1,r}}^{-1} \times u_{1,r}$$
$$q_r = (x_{1,r}v_{1,r} \times {u_{1,r}}^{-1} + x_{2,r}v_{2,r} \times {u_{1,r}}^{-1} + x_{3,r}v_{3,r} \times {u_{1,r}}^{-1}) \times u_{1,r}$$
$$v_{1,r} \times {u_{1,r}}^{-1} = e^{2k\pi v_{2,i}} \times e^{-2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}$$
$$v_{1,r} \times {u_{1,r}}^{-1} = \otimes_e^r\left(e^{2k\pi v_{2,i}}\,;\, e^{-2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}\right)$$

We must know that: $e^{-2k\pi(v_{1,i}+v_{2,i}+v_{3,i})} = e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}$

$$v_{1,r} \times {u_{1,r}}^{-1} = \otimes_e^r\left(e^{2k\pi v_{2,i}}\,;\, e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}\right)$$
$$v_{1,r} \times {u_{1,r}}^{-1} = T \circ R_{\left(O\,;\,-\frac{\pi}{2}\right)}\left[\Pi_i \times \otimes_e^i\left(e^{\frac{\pi}{2}v_{2,i}}\,;\, e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}\right)\right]$$
$$v_{1,r} \times {u_{1,r}}^{-1} = T \circ R_{\left(O\,;\,-\frac{\pi}{2}\right)}\left[\Pi_i \times e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}+v_{3,i})}\right]$$
$$v_{1,r} \times {u_{1,r}}^{-1} = T \circ R_{\left(O\,;\,-\frac{\pi}{2}\right)}\left[(-) \times (-)e^{\frac{\pi}{2}(v_{1,i}+v_{3,i})}\right]$$
$$v_{1,r} \times {u_{1,r}}^{-1} = T \circ R_{\left(O\,;\,-\frac{\pi}{2}\right)}\left[e^{\frac{\pi}{2}(v_{1,i}+v_{3,i})}\right]$$
$$v_{1,r} \times {u_{1,r}}^{-1} = e^{2k\pi(v_{1,i}+v_{3,i})}$$
$$v_{1,r} \times {u_{1,r}}^{-1} = s_{3,r}$$

In the same way, we obtain:

$$v_{2,r} \times {u_{1,r}}^{-1} = \otimes_e^r\left(e^{2k\pi v_{3,i}} \times e^{-2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}\right)$$
$$v_{2,r} \times {u_{1,r}}^{-1} = \otimes_e^r\left(e^{2k\pi v_{3,i}} \times e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}\right)$$
$$v_{2,r} \times {u_{1,r}}^{-1} = T \circ R_{\left(O\,;\,-\frac{\pi}{2}\right)}\left[\Pi_i \times \otimes_e^i\left(e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}\right)\right]$$
$$v_{2,r} \times {u_{1,r}}^{-1} = s_{1,r}$$
$$v_{3,r} \times {u_{1,r}}^{-1} = \otimes_e^r\left(e^{2k\pi v_{1,i}} \times e^{-2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}\right)$$
$$v_{3,r} \times {u_{1,r}}^{-1} = \otimes_e^r\left(e^{2k\pi v_{1,i}} \times e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}\right)$$
$$v_{3,r} \times {u_{1,r}}^{-1} = T \circ R_{\left(O\,;\,-\frac{\pi}{2}\right)}\left[\Pi_i \times \otimes_e^i\left(e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}\right)\right]$$
$$v_{3,r} \times {u_{1,r}}^{-1} = s_{2,r}$$

Whence $q_r = (x_{1,r}s_{3,r} + x_{2,r}s_{1,r} + x_{3,r}s_{2,r}) \times u_{1,r}$

$q_r = (x_{2,r}s_{1,r} + x_{3,r}s_{2,r} + x_{1,r}s_{3,r}) \times u_{1,r}$

We can clearly see that $x_{2,r}s_{1,r} + x_{3,r}s_{2,r} + x_{1,r}s_{3,r}$ is a linear combination with real coefficients and real bases.

If we designate by the number $a_{1,r}$ this linear combination with real coefficients and real bases of dimension 3 such that: $a_{1,r} = x_{2,r}s_{1,r} + x_{3,r}s_{2,r} + x_{1,r}s_{3,r}$

So $q_r = a_{1,r}u_{1,r}$

Likewise, we have the equality:

$u_{2,i} = v_{1,i} \times v_{2,i} \times v_{3,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}v_{3,i}} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}$

The number $u_{2,i}$ designates imaginary base of module $\rho = \sqrt{3}$.

So: $q_i = q_i \times u_{2,i}{}^{-1} \times u_{2,i}$

$q_i = (y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}) \times u_{2,i}{}^{-1} \times u_{2,i}$

$q_i = (y_{1,i}v_{1,i} \times u_{2,i}{}^{-1} + y_{2,i}v_{2,i} \times u_{2,i}{}^{-1} + y_{3,i}v_{3,i} \times u_{2,i}{}^{-1}) \times u_{2,i}$

$v_{1,i} \times u_{2,i}{}^{-1} = e^{\frac{\pi}{2}v_{1,i}} \times e^{-\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{-\frac{\pi}{2}(v_{2,i}+v_{3,i})} = -s_{3,i}$

If we use the imaginary product; we obtain:

$v_{1,i} \times u_{2,i}{}^{-1} = \otimes_e^i \left(e^{\frac{\pi}{2}v_{1,i}} \times e^{-\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}\right)$

$v_{1,i} \times u_{2,i}{}^{-1} = e^{\frac{\pi}{2}(0v_{1,i}-v_{2,i}-v_{3,i})}$

$v_{1,i} \times u_{2,i}{}^{-1} = e^{\frac{\pi}{2}(-v_{2,i}-v_{3,i})}$

$v_{1,i} \times u_{2,i}{}^{-1} = -s_{3,i}$

In the same way, we obtain:

$v_{2,i} \times u_{2,i}{}^{-1} = e^{\frac{\pi}{2}v_{2,i}} \times e^{-\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(-v_{1,i}-v_{3,i})} = -s_{1,i}$

$v_{3,i} \times u_{2,i}{}^{-1} = e^{\frac{\pi}{2}v_{3,i}} \times e^{-\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(-v_{1,i}-v_{2,i})} = -s_{2,i}$

Whence: $q_i = (-y_{1,i}s_{3,i} - y_{2,i}s_{1,i} - y_{3,i}s_{2,i}) \times u_{2,i}$

$q_i = (-y_{2,i}s_{1,i} - y_{3,i}s_{2,i} - y_{1,i}s_{3,i}) \times u_{2,i}$

We can clearly see that $-y_{2,i}s_{1,i} - y_{3,i}s_{2,i} - y_{1,i}s_{3,i}$ is a linear combination with real coefficients and imaginary bases.

If we designate by the number $b_{2,i}$ this linear combination with real coefficients and imaginary bases of dimension 3 such that: $b_{2,i} = -y_{2,i}s_{1,i} - y_{3,i}s_{2,i} - y_{1,i}s_{3,i}$

So $q_i = b_{2,i}u_{2,i}$

Therefore: $h = a_{1,r}u_{1,r} + b_{2,i}u_{2,i}$

All this shows that the hyper-complex numbers h of the set $\mathbb{S}_{(3;3)}$ can be written in a plane complex form:

$h = a_{1,r}u_{1,r} + b_{2,i}u_{2,i}$;

$a_{1,r}$ a linear combination with real coefficients and real bases of dimension 3.

$b_{2,i}$ a linear combination with real coefficients and imaginary bases of dimension 3.

Thus we obtain the hyper-plane $P_{(3;3)}$ associated with the hyper-space $E_{(3;3)}$.

So we can associate the hyper-complex number writing $h = a_{1,r}u_{1,r} + b_{2,i}u_{2,i}$ of dimension 6 the following multiplication table:

$\times$	$u_{1,r}$	$u_{2,i}$
$u_{1,r}$	$u_{1,r}$	$u_{2,i}$
$u_{2,i}$	$u_{2,i}$	$-u_{1,r}$

Table 5: Multiplication table of upper base $\{u_{1,r}\ ;\ u_{2,i}\}$

Thus, the rules of multiplication in $\mathbb{S}_{(3;3)}$ are the same as those in $\mathbb{S}_{(1;1)}$. Consequently the operation $\times$ of the multiplication associated with the Euler products $\otimes_e^r$ and $\otimes_e^i$ in the set $\mathbb{S}_{(3;3)}$ is a commutative, associative and bilinear operation and admits as element neutral $u_{1,r} = e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}$.

We can conclude that $\mathbb{S}_{(3;3)}$ equipped with the internal operations $+$ and $\times$ is an associative, commutative, unitary and bilinear algebra on the hyper-complex number set $\mathbb{S}_{(3;3)}$.

For a hyper-complex number $h = a_{1,r}u_{1,r} + b_{2,i}u_{2,i}$ of dimension 3; its corresponding direct similarity matrix is written as follows:

$$\mathcal{M}_2(h) = \begin{pmatrix} a_{1,r} & -b_{2,i} \\ b_{2,i} & a_{1,r} \end{pmatrix}$$

3.8 Subsets of $\mathbb{S}_{(3;3)}$

The set $\mathbb{S}_{(3;3)}$ is a superimposed set with 3 real and imaginary dimensions. The subsets of $\mathbb{S}_{(3;3)}$ are superimposed sets with real and imaginary dimensions less than or equal to 3. Thus we distinguish the subsets:

Of type $\mathbb{S}_{(m;p)}$ with $m \leq 3$ and $p \leq 3$. There are $2^4 - 1$ subsets for these types.

Of type $\mathbb{S}_{(3;3)\backslash(m;p)}$ the subsets of $\mathbb{S}_{(3;3)}$ deprived of the bases of $\mathbb{S}_{(m;p)}$. There are $2^4 - 2$ subsets for these types.

To this must be added the subsets of $\mathbb{S}_{(3;3)}$ deprived of some real or imaginary bases

3.9 Study of the case where the primary real bases are superimposed on the same axis

We know that, all real bases (primary, intermediate and upper) have the same module $\rho_r = 1$. Therefore, we can study the case where the primary real bases are superimposed; by asking:

$v_{1,r} = v_{2,r} = v_{3,r} = v_r$

We will then have for any "superimposed hyper-complex":

$h = x_{1,r}v_{1,r} + x_{2,r}v_{2,r} + x_{3,r}v_{3,r} + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

$h = x_{1,r}v_r + x_{2,r}v_r + x_{3,r}v_r + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

$h = \left(x_{1,r} + x_{2,r} + x_{3,r}\right)v_r + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

If we pose that: $x_{1,r} + x_{2,r} + x_{3,r} = x_r$

We obtain the equality $h = x_r v_r + y_{1,i}v_{1,i} + y_{2,i}v_{2,i} + y_{3,i}v_{3,i}$

This expression of the "superimposed hyper-complex" h is given with the primary complex bases. Geometrically, we can say that; v_r is the real base carried by the unique hyper-axis defined as, the unique axis perpendicular to the imaginary hyper-space $E_{(0;3)} = \{(v_{1,i}\ ;\ v_{2,i}\ ;\ v_{3,i})\}$. Which corresponds to the subset $\mathbb{S}_{(1;3)}$.

However, we know that the three real bases hide intermediate and upper bases. So we can associate; any "superimposed hyper-complex" h in the subset $\mathbb{S}_{(1;3)}$; for its general expression the following multiplication table:

$\times$	v_r	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$	$s_{1,i}$	$s_{2,i}$	$s_{3,i}$	$s_{4,i}$	$s_{5,i}$	$s_{6,i}$	$u_{1,i}$	$u_{2,i}$	$u_{3,i}$	$u_{4,i}$
v_r	v_r	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$	$s_{1,i}$	$s_{2,i}$	$s_{3,i}$	$s_{4,i}$	$s_{5,i}$	$s_{6,i}$	$u_{1,i}$	$u_{2,i}$	$u_{3,i}$	$u_{4,i}$
$v_{1,i}$	$v_{1,i}$	$-v_r$	$s_{2,i}$	$s_{1,i}$	$-v_{3,i}$	$-v_{2,i}$	$u_{2,i}$	$v_{3,i}$	$v_{2,i}$	$u_{3,i}$	$s_{6,i}$	$-s_{3,i}$	$-s_{6,i}$	$s_{3,i}$
$v_{2,i}$	$v_{2,i}$	$s_{2,i}$	$-v_r$	$s_{3,i}$	$u_{2,i}$	$-v_{1,i}$	$-v_{3,i}$	$u_{4,i}$	$v_{1,i}$	$v_{3,i}$	$s_{1,i}$	$-s_{1,i}$	$s_{4,i}$	$-s_{4,i}$
$v_{3,i}$	$v_{3,i}$	$s_{1,i}$	$s_{3,i}$	$-v_r$	$-v_{1,i}$	$u_{2,i}$	$-v_{2,i}$	$v_{1,i}$	$u_{1,i}$	$v_{2,i}$	$-s_{5,i}$	$-s_{2,i}$	$s_{2,i}$	$s_{5,i}$
$s_{1,i}$	$s_{1,i}$	$-v_{3,i}$	$u_{2,i}$	$-v_{1,i}$	$-v_r$	$-s_{3,i}$	$-s_{2,i}$	$-v_r$	$s_{6,i}$	$s_{2,i}$	$v_{2,i}$	$-v_{2,i}$	$-v_{2,i}$	$-v_{2,i}$
$s_{2,i}$	$s_{2,i}$	$-v_{2,i}$	$-v_{1,i}$	$u_{2,i}$	$-s_{3,i}$	$-v_r$	$-s_{1,r}$	$s_{3,i}$	$-v_r$	$s_{4,i}$	$-v_{3,i}$	$-v_{3,i}$	$v_{3,i}$	$-v_{3,i}$
$s_{3,i}$	$s_{3,i}$	$u_{2,i}$	$-v_{3,i}$	$-v_{2,i}$	$-s_{2,i}$	$-s_{1,r}$	$-v_r$	$s_{5,i}$	$s_{1,i}$	$-v_r$	$-v_{1,i}$	$-v_{1,i}$	$-v_{1,i}$	$v_{1,i}$
$s_{4,i}$	$s_{4,i}$	$v_{3,i}$	$u_{4,i}$	$v_{1,i}$	$-v_r$	$s_{3,i}$	$s_{5,i}$	$-v_r$	$-s_{6,i}$	$-s_{5,i}$	$v_{2,i}$	$-v_{2,i}$	$v_{2,i}$	$-v_{2,i}$
$s_{5,i}$	$s_{5,i}$	$v_{2,i}$	$v_{1,i}$	$u_{1,i}$	$s_{6,i}$	$-v_r$	$s_{1,i}$	$-s_{6,i}$	$-v_r$	$-s_{4,i}$	$-v_{3,i}$	$-v_{3,i}$	$v_{3,i}$	$v_{3,i}$
$s_{6,i}$	$s_{6,i}$	$u_{3,i}$	$v_{3,i}$	$v_{2,i}$	$s_{2,i}$	$s_{4,i}$	$-v_r$	$-s_{5,i}$	$-s_{4,i}$	$-v_r$	$v_{1,i}$	$-v_{1,i}$	$-v_{1,i}$	$v_{1,i}$
$u_{1,i}$	$u_{1,i}$	$s_{6,i}$	$s_{1,i}$	$-s_{5,i}$	$v_{2,i}$	$-v_{3,i}$	$-v_{1,i}$	$v_{2,i}$	$-v_{3,i}$	$v_{1,i}$	$-v_r$	$-v_r$	$-v_r$	$-v_r$
$u_{2,i}$	$u_{2,i}$	$-s_{3,i}$	$-s_{1,i}$	$-s_{2,i}$	$-v_{2,i}$	$-v_{3,i}$	$-v_{1,i}$	$-v_{2,i}$	$-v_{3,i}$	$-v_{1,i}$	$-v_r$	$-v_r$	$-v_r$	$-v_r$
$u_{3,i}$	$u_{3,i}$	$-s_{6,i}$	$s_{4,i}$	$s_{2,i}$	$-v_{2,i}$	$v_{3,i}$	$-v_{1,i}$	$v_{2,i}$	$v_{3,i}$	$-v_{1,i}$	$-v_r$	$-v_r$	$-v_r$	$-v_r$
$u_{4,i}$	$u_{4,i}$	$s_{3,i}$	$-s_{4,i}$	$s_{5,i}$	$-v_{2,i}$	$-v_{3,i}$	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$	$v_{1,i}$	$-v_r$	$-v_r$	$-v_r$	$-v_r$

Table 6: multiplication table of subset $\mathbb{S}_{(1;3)}$ when $v_{1,r} = v_{2,r} = v_{3,r} = v_r$

The data used to establish table 6 are listed in the appendices on page 25.

3.10 Extension of set $\mathbb{C}$ by superposition of two complex planes

Let us consider the complex planes $(O\ ;\ v_r\ ;\ v_i)$ and $(O\ ;\ v_r\ ;\ v_j)$ defined by:

- Such that any number $s_1 \in \mathbb{C}$; affix of a point M_1 of the complex plane $(O\ ;\ v_r\ ;\ v_i)$; we have: $s_1 = a + yi$ (with $a \in \mathbb{R}$; $y \in \mathbb{R}$ and i an imaginary number such that: $i^2 = -1$).

  We associate with this plan the complex base $\{1; i\}$; where $i = e^{\frac{\pi}{2}i}$.
- Such that any number $s_2 \in \mathbb{C}$; affix of a point M_2 of the complex plane $(O\ ;\ v_r\ ;\ v_j)$; we have : $s_2 = b + zj$ (with $b \in \mathbb{R}$; $z \in \mathbb{R}$ and j an imaginary number such that: $j^2 = -1$)

  We associate with this plan the complex base $\{1; j\}$; where $j = e^{\frac{\pi}{2}j}$.

If we superimpose the complex planes $(O\ ;\ v_r\ ;\ v_i)$ and $(O\ ;\ v_r\ ;\ v_j)$ orthogonally; we obtain the superimposed complex space $(O\ ;\ v_r\ ;\ v_i\ ;\ v_j)$. Thus there exists a superimposed hyper-complex number s such that: $s = s_1 + s_2$

We then have: $s = a + yi + b + zj$

$s = a + b + yi + zj$; if we put $x = a + b$ we will have.

$s = x + yi + zj$

We can therefore define the three-dimensional set $\mathbb{C}_{(1;2)}$ as an extension of set $\mathbb{C}$ by orthogonal superposition of two complex planes. In this configuration the real axis $(O\ ;\ v_r)$ is perpendicular to the imaginary plane $(O\ ;\ v_i\ ;\ v_j)$.
To the set $\mathbb{C}_{(1;2)}$; we can associate the following multiplication table.

$\times$	1	$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i-j)}$
1	1	$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i-j)}$
$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}i}$	-1	$e^{\frac{\pi}{2}(i+j)}$	$-e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}j}$
$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	-1	$-e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}i}$
$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i+j)}$	$-e^{\frac{\pi}{2}j}$	$-e^{\frac{\pi}{2}i}$	-1	-1
$e^{\frac{\pi}{2}(i-j)}$	$e^{\frac{\pi}{2}(i-j)}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}i}$	-1	-1

Table 7: multiplication table of $\mathbb{C}_{(1;2)}$

The data used to establish table 7 are listed in the appendices on page 28.

Note: $e^{\frac{\pi}{2}(j-i)} = -e^{\frac{\pi}{2}(i-j)}$; so no need to put $e^{\frac{\pi}{2}(j-i)}$ in the table.

4 Sets $\mathbb{S}_{(9;9)}$; …….. ; $\mathbb{S}_{(n;n)}$; $\mathbb{S}_{(3n;3n)}$

4.1 Algebraic space $\mathbb{S}_{(9;9)}$

4.1.1 Construction by orthogonal superposition.

Intuitively, it is difficult to visualize the addition of the 4th dimension and subsequent dimensions. However, we can obtain the algebraic space $\mathbb{S}_{(9;9)}$ by an orthogonal superposition of three hyper-planes of dimension 3: $P_{(3;3)}$; $P'_{(3;3)}$ and $P''_{(3;3)}$.
$P_{(3;3)}$: Hyper-complex plan associated with real bases $\{v_{1,r}\ ;v_{2,r}\ ;v_{3,r}\}$ and imaginary bases $\{v_{1,i}\ ;\ v_{2,i}\ ;\ v_{3,i}\}$; of upper base $\{u_{1,r}\ ;u_{2,i}\}$.
$P'_{(3;3)}$: Hyper-complex plane associated with real bases $\{v_{4,r}\ ;v_{5,r}\ ;v_{6,r}\}$ and imaginary bases $\{v_{4,i}\ ;\ v_{5,i}\ ;\ v_{6,i}\}$; upper basis $\{u_{2,r}\ ;u_{3,i}\}$.
$P''_{(3;3)}$: Hyper-complex plane associated with real bases $\{v_{7,r}\ ;v_{8,r}\ ;v_{9,r}\}$ and imaginary bases $\{v_{7,i}\ ;\ v_{8,i}\ ;\ v_{9,i}\}$; upper base $\{u_{3,r}\ ;u_{1,i}\}$
The orthogonal superposition of the hyper-planes $P_{(3;3)}$; $P'_{(3;3)}$ and $P''_{(3;3)}$ in their rotation orders $P_{(3;3)} \longrightarrow P'_{(3;3)} \longrightarrow P''_{(3;3)} \longrightarrow P_{(3;3)}$; gives us a geometric hyper-space:
$E_{(9;9)} = P_{(3;3)} \perp P'_{(3;3)} \perp P''_{(3;3)}$.
Considering $h_1 = a_{1,r}u_{1,r} + b_{2,i}u_{2,i}$ the affix of a hyper-point of the plane $P_{(3;3)}$

$a_{1,r}$ a linear combination with real coefficients and real bases
$b_{2,i}$ a linear combination with real coefficients and imaginary bases
$u_{1,r} = e^{2k\pi(v_{1,i}+v_{2,i}+v_{3,i})}$ and $u_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}$

If $h_2 = a_{2,r}u_{2,r} + b_{3,i}u_{3,i}$ is the affix of a hyper-point of the plane $P'_{(3;3)}$

$a_{2,r}$ a linear combination with real coefficients and real bases
$b_{3,i}$ a linear combination with real coefficients and imaginary bases
$u_{2,r} = e^{2k\pi(v_{4,i}+v_{5,i}+v_{6,i})}$ and $u_{3,i} = e^{\frac{\pi}{2}(v_{4,i}+v_{5,i}+v_{6,i})}$

If $h_3 = a_{3,r}u_{3,r} + b_{1,i}u_{1,i}$ is the affix of a hyper-point of the plane $P''_{(3;3)}$

$a_{3,r}$ a linear combination with real coefficients and real bases
$b_{1,i}$ a linear combination with real coefficients and imaginary bases
$u_{2,r} = e^{2k\pi(v_{7,i}+v_{8,i}+v_{9,i})}$ and $u_{3,i} = e^{\frac{\pi}{2}(v_{7,i}+v_{8,i}+v_{9,i})}$

Then at any point M in the space, $E_{(9;9)}$ we can associate a number $h = h_1 + h_2 + h_3$
We therefore have: $h = (a_{1,r}u_{1,r} + b_{2,i}u_{2,i}) + (a_{2,r}u_{2,r} + b_{3,i}u_{3,i}) + (a_{3,r}u_{3,r} + b_{1,i}u_{1,i})$

$$h = a_{1,r}u_{1,r} + a_{2,r}u_{2,r} + a_{3,r}u_{3,r} + b_{1,i}u_{1,i} + b_{2,i}u_{2,i} + b_{3,i}u_{3,i}$$

The hyper-complex writing of the number h of dimension 18 includes two parts:
A real part $q_r = a_{1,r}u_{1,r} + a_{2,r}u_{2,r} + a_{3,r}u_{3,r}$
An imaginary part $q_i = b_{1,i}u_{1,i} + b_{2,i}u_{2,i} + b_{3,i}u_{3,i}$
$E_{(9;9)}$ is therefore the superposition of a hyper-space with 9 real dimensions and a hyper-space with 9 imaginary dimensions:
$\mathfrak{B}_{9,r} = \{v_{1,r} ; v_{2,r} ; \dots ; v_{9,r}\}$ and $\mathfrak{B}_{9,i} = \{v_{1,i} ; v_{2,i} ; \dots ; v_{9,i}\}$
We note: $E_{(9;9)} = \{(v_{1,r} ; v_{2,r} ; \dots ; v_{9,r}) ; (v_{1,i} ; v_{2,i} ; \dots ; v_{9,i})\}$
Thus the upper base of the set $\mathbb{S}_{(9;9)}$ is: $\begin{cases} e^{2k\pi[\sum_{p=1}^{9}(v_{p,i})]} \\ e^{\frac{\pi}{2}[\sum_{p=1}^{9}(v_{p,i})]} \end{cases}$ with $k = 0$

4.1.2 Hyper-plane $P_{(9;9)}$

The geometric and orthogonal superposition of the planes $P_{(3;3)}$; $P'_{(3;3)}$ and $P''_{(3;3)}$ gave us the hyperspace $E_{(9;9)}$ associated with the set $\mathbb{S}_{(9;9)}$ has 9 real dimensions and 9 imaginary dimensions.

If we note ${}^{\sqrt{9}}_{\rho}V_{1,r} = e^{2k\pi[\sum_{p=1}^{9}(v_{p,r})]}$ and ${}^{\sqrt{9}}_{\rho}V_{2,i} = e^{\frac{\pi}{2}[\sum_{p=1}^{9}(v_{p,i})]}$ the upper bases of the set $\mathbb{S}_{(9;9)}$ and of module $\rho_9 = \sqrt{9} = 3$; then for any number $h \in \mathbb{S}_{(9;9)}$, there exist two numbers $X_{1,r}$ and $Y_{2,i}$ such that:
$q_r = a_{1,r}u_{1,r} + a_{2,r}u_{2,r} + a_{3,r}u_{3,r} = X_{1,r}\,{}^{\sqrt{9}}_{\rho}V_{1,r}$
$q_i = b_{1,i}u_{1,i} + b_{2,i}u_{2,i} + b_{3,i}u_{3,i} = Y_{2,i}\,{}^{\sqrt{9}}_{\rho}V_{2,i}$
So we have: $h = X_{1,r}\,{}^{\sqrt{9}}_{\rho}V_{1,r} + Y_{2,i}\,{}^{\sqrt{9}}_{\rho}V_{2,i}$
$X_{1,r}$ a linear combination with real coefficients and real bases of dimension 9 and module 1 . If we denote ${}^{\sqrt{6}}_{\rho}S_{p,r}$ the intermediate real bases of module 1 then: $X_{1,r} = \sum_{p=1}^{3}(a_{p,r}\,{}^{\sqrt{6}}_{\rho}S_{p,r})$ with $p \in \{1 ; 2 ; 3\}$ and $a_{p,r}$ the real coefficients of h.
$Y_{2,i}$ a linear combination with real coefficients and imaginary bases of dimension 9 and module $\rho_6 = \sqrt{6}$. If we note ${}^{\sqrt{6}}_{\rho}S_{n,i}$ the intermediate imaginary bases of module $\rho_6 = \sqrt{6}$ then: $Y_{2,i} = \sum_{p=1}^{3} b_{p,i}\,{}^{\sqrt{6}}_{\rho}S_{p,i}$ with $p \in \{1 ; 2 ; 3\}$ and $b_{p,i}$ the imaginary coefficients of h.
Thus we obtain a complex hyper-plane $P_{(9;9)}$ with 9 dimensions of bases greater than $\{{}^{\sqrt{9}}_{\rho}V_{1,r} ; {}^{\sqrt{9}}_{\rho}V_{2,i}\}$.
We can also write the numbers $X_{1,r}$ and $Y_{2,i}$ as a linear combination respectively of the 9 intermediate real bases of module 1 and the 9 intermediate imaginary bases of module 1.
Thus $X_{1,r} = \sum_{p=1}^{9}(a_{p,r}\,{}^{\sqrt{8}}_{\rho}S_{p,r})$ with $a_{p,r}$ the coefficients of the initial real bases and ${}^{\sqrt{8}}_{\rho}S_{p,r}$ the intermediate bases of module 1.
And $Y_{2,i} = \sum_{p=1}^{9}(b_{p,i}\,{}^{\sqrt{8}}_{\rho}S_{p,i})$ with $b_{p,i}$ the coefficients of the initial imaginary bases and ${}^{\sqrt{8}}_{\rho}S_{p,i}$ the intermediate bases of imaginary module $\rho_8 = \sqrt{8}$.

4.1.3 Subsets of $\mathbb{S}_{(9;9)}$

In addition to the set $\mathbb{S}_{(3;3)}$ and its subsets we distinguish the subsets $\mathbb{S}_{(m;l)}$ with $3 \le m \le 9$ and $3 \le l \le 9$ such that: if $m = 9$ then $l < 9$ and if $l = 9$ then $m < 9$.
The upper bases of the subsets $\mathbb{S}_{(m;l)}$ are respectively:
${}^{\sqrt{m}}_{\rho}V_{1,r} = e^{2k\pi[\sum_{p=1}^{m}(v_{p,r})]}$: real base
${}^{\sqrt{l}}_{\rho}V_{2,i} = e^{\frac{\pi}{2}[\sum_{p=1}^{l}(v_{p,i})]}$: Imaginary base
Thus the set $\mathbb{S}_{(9;9)}$ inherits the algebraic properties of the set $\mathbb{S}_{(3;3)}$. This construction carried out by geometric superposition gives the set $\mathbb{S}_{(9;9)}$ an algebraic structure of dimension 18. Consequently $\mathbb{S}_{(9;9)}$ is a $\mathbb{C}$ -algebra.

4.2 Algebraic space $\mathbb{S}_{(3n;3n)}$

4.2.1 Complete $\mathbb{C}$ -algebra

Considering $\mathbb{S}_{(n;n)}$ and $\mathbb{S}_{(p;p)}$ two $\mathbb{C}$-algebra of dimensions n and p. We can say that:
$\mathbb{S}_{(n;n)}$ is a complete $\mathbb{C}$ -algebra if $n = 3^k$ with $k \in \mathbb{N}$
$\mathbb{S}_{(p;p)}$ is a complete $\mathbb{C}$ -algebra generated by the complete $\mathbb{C}$ -algebra $\mathbb{S}_{(n;n)}$ if $p = 3n$.

4.3 The complete $\mathbb{C}$-algèbre $\mathbb{S}_{(3n;3n)}$

If $P_{(n;n)}$ is a hyper-plane associated with $\mathbb{S}_{(n;n)}$ a complete $\mathbb{C}$-algèbre in $2n$ dimensions, with real and imaginary bases $\{v_{1,r} ; v_{2,r} ; \dots ; v_{n,r}\}$; $\{v_{1,i} ; v_{2,i} ; \dots ; v_{n,i}\}$, then there exist two other hyper-planes $P'_{(n;n)}$ associated with $\mathbb{S}'_{(n;n)}$ a $2n$ dimensional $\mathbb{C}$-algebra, with real and imaginary bases $\{v_{(n+1),r} ; v_{(n+2),r} ; \dots ; v_{2n,r}\}$; $\{v_{(n+1),i} ; v_{(n+2),i} ; \dots ; v_{2n,i}\}$ and $P''_{(n;n)}$ associated with $\mathbb{S}''_{(n;n)}$ a $2n$ dimensional $\mathbb{C}$-algebra, with real and imaginary bases $\{v_{(2n+1),r} ; v_{(2n+2),r} ; \dots ; v_{3n,r}\}$;

$\{v_{(2n+1),i}\ ;\ v_{(2n+2),i}\ ;\ \dots\dots\dots\ ;\ v_{3n,i}\}$.

Under these conditions the orthogonal superposition of the hyper-planes $P_{(n;n)}$; $P'_{(n;n)}$ and $P''_{(n;n)}$ gives us a hyperspace $E_{(3n;3n)} = P_{(n;n)} \perp P'_{(n;n)} \perp P''_{(n;n)}$ noted:

$E_{(3n;3n)} = \{(v_{1,r}\ ; v_{2,r}\ ; \dots\dots\dots\dots\ ; v_{3n,r})\ ; (v_{1,i}\ ;\ v_{2,i}\ ; \dots\dots\dots\dots\ ;\ v_{3n,i})\}$.

Consequently, the set $\mathbb{S}_{(3n;3n)}$ is a complete $\mathbb{C}$-algèbre generated by the complete $\mathbb{C}$-algèbre $\mathbb{S}_{(n;n)}$ with higher bases: ${}^{\sqrt{3n}}_{\rho}V_{1,r} = e^{2k\pi[\sum_{p=1}^{3n}(v_{p,i})]}$ and ${}^{\sqrt{3n}}_{\rho}V_{2,i} = e^{\frac{\pi}{2}[\sum_{p=1}^{3n}(v_{p,i})]}$ respectively the real basis and imaginary basis (where 2n is the dimension of $\mathbb{S}_{(n;n)}$).

4.3.1 Hyper-planes $P_{(3n;\,3n)}$

The geometric and orthogonal superposition of the planes $P_{(n;n)}$; $P'_{(n;n)}$ and $P''_{(n;n)}$ gave us the hyperspace $E_{(3n;3n)}$ associated with the set $\mathbb{S}_{(3n;3n)}$ to $3n$ real and imaginary dimensions.

If $h_1 = X_{1,r}\,{}^{\sqrt{n}}_{\rho}V_{1,r} + Y_{2,i}\,{}^{\sqrt{n}}_{\rho}V_{2,i}$ is the affix of a hyper-point of the plane $P_{(n;n)}$

$X_{1,r}$ a linear combination with real coefficients and real bases

$Y_{2,i}$ a linear combination with real coefficients and imaginary bases

${}^{\sqrt{n}}_{\rho}V_{1,r} = e^{2k\pi[\sum_{p=1}^{n}(v_{p,r})]}$ and ${}^{\sqrt{n}}_{\rho}V_{2,i} = e^{\frac{\pi}{2}[\sum_{p=1}^{n}(v_{p,i})]}$

If $h_2 = Y_{1,r}\,{}^{\sqrt{n}}_{\rho}V_{2,r} + Z_{2,i}\,{}^{\sqrt{n}}_{\rho}V_{3,i}$ is the affix of a hyper-point of the plane $P'_{(n;n)}$

$Y_{1,r}$ a linear combination with n real coefficients and n real bases

$Z_{2,i}$ a linear combination with n real coefficients and n imaginary bases

${}^{\sqrt{n}}_{\rho}V_{2,r} = e^{2k\pi[\sum_{p=n+1}^{2n}(v_{p,r})]}$ and ${}^{\sqrt{n}}_{\rho}V_{3,i} = e^{\frac{\pi}{2}[\sum_{p=n+1}^{2n}(v_{p,i})]}$

If $h_3 = Z_{1,r}\,{}^{\sqrt{n}}_{\rho}V_{3,r} + X_{2,i}\,{}^{\sqrt{n}}_{\rho}V_{1,i}$ is the affix of a hyper-point of the plane $P''_{(n;n)}$

$Z_{1,r}$ a linear combination with n real coefficients and n real bases

$X_{2,i}$ a linear combination with n real coefficients and n imaginary bases

${}^{\sqrt{n}}_{\rho}V_{3,r} = e^{2k\pi[\sum_{p=2n+1}^{3n}(v_{p,r})]}$ and ${}^{\sqrt{n}}_{\rho}V_{1,i} = e^{\frac{\pi}{2}[\sum_{p=3n+1}^{3n}(v_{p,i})]}$

Then at any point Min the space, $E_{(3n;3n)}$ we can associate a number $h = h_1 + h_2 + h_3$

Therefore, we have:

$h = \left(X_{1,r}\,{}^{\sqrt{n}}_{\rho}V_{1,r} + Y_{2,i}\,{}^{\sqrt{n}}_{\rho}V_{2,i}\right) + \left(X_{2,r}\,{}^{\sqrt{n}}_{\rho}V_{2,r} + Y_{3,i}\,{}^{\sqrt{n}}_{\rho}V_{3,i}\right) + \left(X_{3,r}\,{}^{\sqrt{n}}_{\rho}V_{3,r} + Y_{1,i}\,{}^{\sqrt{n}}_{\rho}V_{1,i}\right)$

$h = X_{1,r}\,{}^{\sqrt{n}}_{\rho}V_{1,r} + X_{2,r}\,{}^{\sqrt{n}}_{\rho}V_{2,r} + X_{3,r}\,{}^{\sqrt{n}}_{\rho}V_{3,r} + Y_{1,i}\,{}^{\sqrt{n}}_{\rho}V_{1,i} + Y_{2,i}\,{}^{\sqrt{n}}_{\rho}V_{2,i} + Y_{3,i}\,{}^{\sqrt{n}}_{\rho}V_{3,i}$

The hyper-complex writing of the number h of dimension $2 \times 3n = 6n$ includes two parts:

A real part $q_r = X_{1,r}\,{}^{\sqrt{n}}_{\rho}V_{1,r} + X_{2,r}\,{}^{\sqrt{n}}_{\rho}V_{2,r} + X_{3,r}\,{}^{\sqrt{n}}_{\rho}V_{3,r}$

An imaginary game $q_i = Y_{1,i}\,{}^{\sqrt{n}}_{\rho}V_{1,i} + Y_{2,i}\,{}^{\sqrt{n}}_{\rho}V_{2,i} + Y_{3,i}\,{}^{\sqrt{n}}_{\rho}V_{3,i}$

Then for any number $h \in \mathbb{S}_{(3n;3n)}$, there exist two numbers $A_{1,r}$ and $B_{2,i}$ such that:

$h = A_{1,r}\,{}^{\sqrt{3n}}_{\rho}V_{1,r} + B_{2,i}\,{}^{\sqrt{3n}}_{\rho}V_{2,i}$

With $A_{1,r}$ a linear combination with real coefficients and real bases of dimension $3n$ and module 1. If we note ${}^{\sqrt{2n}}_{\rho}S_{p,r}$ the intermediate real bases of module 1 , then: $A_{1,r} = \sum_{p=1}^{3}(X_{p,r}\,{}^{\sqrt{2n}}_{\rho}S_{p,r})$ with $p \in \{1\,;2\,;3\}$ and $X_{p,r}$; the real coefficients of any number $h \in \mathbb{S}_{(3n;3n)}$.

With $B_{2,i}$ a linear combination with real coefficients and imaginary bases of dimension 3n and imaginary module $\rho_{2n} = \sqrt{2n}$. If we note ${}^{\sqrt{2n}}_{\rho}S_{p,i}$ the intermediate imaginary bases of module $\rho_{2n} = \sqrt{2n}$ then: $B_{2,i} = \sum_{p=1}^{3} Y_{p,i}\,{}^{\sqrt{2n}}_{\rho}S_{p,i}$ with $p \in \{1\,;2\,;3\}$ and $Y_{p,i}$; the imaginary coefficients for any number $h \in \mathbb{S}_{(3n;3n)}$.

Thus we obtain a complex hyper-plane $P_{(3n;3n)}$ with 6n dimensions of bases greater than $\{{}^{\sqrt{3n}}_{\rho}V_{1,r}\ ;\ {}^{\sqrt{3n}}_{\rho}V_{2,i}\}$ of real module 1 and imaginary module $\rho_{3n} = \sqrt{3n}$.

We can also obtain the hyper-plane with the numbers $A_{1,r}$ and $B_{2,i}$ from linear combinations respectively with the $(2n-1)th$ real module bases 1 and imaginary module bases $\rho_{2n-1} = \sqrt{2n-1}$

4.3.2 Subsets of $\mathbb{S}_{(3n;\,3n)}$

In addition to the set $\mathbb{S}_{(n;n)}$ and its subsets we distinguish the subsets $\mathbb{S}_{(m;l)}$ with $n \le m \le 3n$ and $n \le l \le 3n$; with special cases: if $m = 3n$ then $l < 3n$ and if $l = 3n$ then $m < 3n$.

The upper bases of the subsets $\mathbb{S}_{(m;l)}$ are respectively:

${}^{\sqrt{m}}_{\rho}V_{1,r} = e^{2k\pi[\sum_{p=1}^{m}(v_{p,r})]}$: real base

$\sqrt[l]{}_{\rho}V_{2,i} = e^{\frac{\pi}{2}[\sum_{p=1}^{l}(v_{p,i})]}$: imaginary base

4.4 Change of orbital and hyper-space

We saw above that the set $\mathbb{S}_{(n;n)}$ (a complete $\mathbb{C}$-algèbre of dimension n) and its subsets all have primary bases; intermediate bases and a higher base of respective modules: $\rho_1 = \sqrt{1}$; $\rho_2 = \sqrt{2}$; $\rho_3 = \sqrt{3}$; ………. ; $\rho_n = \sqrt{n}$. Which is already consistent with the non-continuous distribution of atomic orbitals.

We know that multiplying bases leads to a change of bases and therefore a discrete variation of their modules.

- For example for the set $\mathbb{S}_{(3;3)}$ if we do the product of the base $v_{1,i}$ by the base $v_{2,i}$ we have:

$v_{1,i} \times v_{2,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = s_{2,i}$

We have a passage from $\rho_1 = \sqrt{1}$ to $\rho_2 = \sqrt{2}$

- Then let's make the product

$v_{3,i} \times s_{2,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = u_{2,i}$

We have a passage from $\rho_2 = \sqrt{2}$ to $\rho_3 = \sqrt{3}$

- And inversely we have

$u_{2,i} \times v_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}v_{2,i}} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}+v_{3,i})} = -e^{\frac{\pi}{2}(v_{1,i}+v_{3,i})} = -s_{1,i}$

We have a transition from $\rho_3 = \sqrt{3}$ to $\rho_2 = \sqrt{2}$

- Likewise

$u_{2,i} \times s_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = e^{\frac{\pi}{2}(2v_{1,i}+2v_{2,i}+v_{3,i})} = -e^{\frac{\pi}{2}v_{3,i}} = -v_{3,i}$

We have a transition from $\rho_3 = \sqrt{3}$ to $\rho_1 = \sqrt{1}$

- Now what about the change of hyper-space?

Let us make the product of

$e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i}+v_{4,i})} = e^{\frac{\pi}{2}(2v_{1,i}+2v_{2,i}+2v_{3,i}+v_{4,i})} = -e^{\frac{\pi}{2}v_{4,i}} = -v_{4,i}$

Which means that we completely leave the set $\mathbb{S}_{(3;3)}$.

All this is like jumping from one orbital to another.

4.5 Hypotheses on the distribution and angular orientation of the intermediate and upper bases

We know that, beyond a dimension $n > 3$; that is to say with the superposition of the three hyper-planes of dimension 6 to obtain the hyper-space of dimension 18, the notion of angles is difficult to visualize and requires the introduction of the notion of hyper-angles. In any case, angles of measurement $\frac{\pi}{4}$ rd appear in the hyper-space of dimension 6. This means a space where there are the angles $\alpha_0 = 2\pi$; $\alpha_1 = \pi$; $\alpha_2 = \frac{\pi}{2}$ and $\alpha_3 = \frac{\pi}{4}$. We notice the appearance of a sequel.

Which leads me to propose the following hypotheses for the angles which gives the orientations of the primary, intermediate and upper bases.

- Hypothesis 1:

$\alpha_0 = \frac{2\pi}{2^0} = 2\pi$; $\alpha_1 = \frac{2\pi}{2^1} = \pi$; $\alpha_3 = \frac{2\pi}{2^2} = \frac{\pi}{2}$; $\alpha_1 = \frac{2\pi}{2^3} = \frac{\pi}{4}$ for dimensions 6 $\mathbb{S}_{(3;3)}$

We can then say that, for a dimension $2n$ we have the angles:

$\alpha_0 = \frac{2\pi}{2^0}$; $\alpha_1 = \frac{2\pi}{2^1}$; $\alpha_2 = \frac{2\pi}{2^2}$; $\alpha_3 = \frac{2\pi}{2^3}$; ……… ; $\alpha_n = \frac{2\pi}{2^k}$

Then we see that the sum of the angles gives the series:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \dots\dots\dots + \alpha_n = \frac{2\pi}{2^0} + \frac{2\pi}{2^1} + \frac{2\pi}{2^2} + \frac{2\pi}{2^3} + \dots\dots\dots + \frac{2\pi}{2^n}$$

If we note $S_{\alpha_k} = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \dots\dots\dots + \alpha_n$

$$S_{\alpha_k} = \sum_{k=0}^{n} \frac{2\pi}{2^n} = 2\pi \sum_{k=0}^{n} \frac{1}{2^n}$$

$$\text{For } n = \infty \text{ then } S_{\alpha_\infty} = 2\pi \sum_{k=0}^{\infty} \frac{1}{2^n} = 2\pi \times \frac{1}{1-\frac{1}{2}} = 4\pi$$

We can clearly see that the sum of the angles which gives the orientations of the primary, intermediate and upper bases is convergent.

- Hypothesis 2:

$$\alpha_1 = \frac{\pi}{2(2-1)} = \frac{\pi}{2}\ ;\ \alpha_2 = \frac{\pi}{2(3-1)} = \frac{\pi}{4}\ ;\ \text{for dimensions } 6\ \ \mathbb{S}_{(3;3)}$$

We can then assume that for a given dimension $2n$ we have the angles:

$$\alpha_1 = \frac{\pi}{2(2-1)} = \frac{\pi}{2}\ ;\ \alpha_2 = \frac{\pi}{2(3-1)} = \frac{\pi}{4}\ \ ;\ \alpha_3 = \frac{\pi}{2(4-1)} = \frac{\pi}{6}\ \ ;\ \dots\dots\dots\ ;\alpha_{n-1} = \frac{\pi}{2(n-1)}$$

Then we see that the sum of the angles gives the series:

$$\alpha_1 + \alpha_2 + \alpha_3 +\ \dots\dots\dots\dots + \alpha_{n-1} = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{6} +\ \dots\dots\dots + \frac{\pi}{2(n-1)}$$

$$\alpha_1 + \alpha_2 + \alpha_3 +\ \dots\dots\dots\dots + \alpha_{n-1} = \frac{\pi}{2}\left(1 + \frac{1}{2} + \frac{1}{3} +\ \dots\dots\dots + \frac{\pi}{(n-1)}\right)$$

$$S_{\alpha_{k-1}} = \frac{\pi}{2}\sum_{p=1}^{n-1}\frac{1}{p} = \frac{\pi}{2}\left(-\frac{1}{n} + \sum_{p=1}^{n}\frac{1}{p}\right)\ ;\ \text{with } k \in \mathbb{N} \text{ and } p \in \mathbb{N}$$

$$\text{For } n = \infty \text{ then }\ \ S_{\alpha_\infty} = \frac{\pi}{2}\sum_{p=1}^{\infty}\frac{1}{p} = \frac{\pi}{2}\zeta(1)$$

We see here that, in this case, the sum of the angles which gives the orientations of the primary, intermediate and upper bases is harmonic.

5 Conclusion

We have just seen that if we use the Euler's formula for complex numbers, we can make an n-dimensional extension of complex numbers. For a 3-dimensional extension with real base 1 and complex bases $i = e^{\frac{\pi}{2}i}$ and $j = e^{\frac{\pi}{2}j}$; we see that the multiplication of complexe numbers $i \times j = j \times i = e^{\frac{\pi}{2}i} \times e^{\frac{\pi}{2}j} = e^{\frac{\pi}{2}j} \times e^{\frac{\pi}{2}i} = e^{\frac{\pi}{2}(i+j)}$ gives us, in a commutative way, a unique complex number $e^{\frac{\pi}{2}(i+j)}$. The complex number $e^{\frac{\pi}{2}(i+j)}$ represents the complex superposition of the complex numbers i and j whose module is equal to $\sqrt{2}$. Still, in this article we see clearly that we can define a commutative algebra on hyper-complex numbers. All this thanks to the work of the illustrious and gread mathematicians such as Leonard Euler, Jérôme Cardan, Jean Robert Argand and that who contributed to making complex numbers a field of study in mathematics. This despite the fact that complex numbers were considered as an artificial mathematical calculation tool by most of comtemporary mathematecians of thier time.

Here we note some analogies with quantum physics. This allow me to ask myself the question: if it is possible to superimpose two electric currents in the same electronic circuit in a under control way? This will allow the development of electronic circuits and transistors capable of superimposing and circulating two different electric currents, in a way to superimposing two binary codes. This offers a manageable approach and mathematical basis for the development of quantum computing as well as quantum telecommunication. Can we have here a calculation basis for the development of nuclear fusion reactor based on hyper-complex superposition fusion of atoms? Likewise, can we see here a calculation basis for supraluminal travel interstellar space by a form of space-time superposition? Knowing that multiplication of hyper-complex numbers allows jumps from one orbital to another. This Would be in accordance with Albert Einstein's theory of relativity; the notion of hyper-plan males it possible to describe a complex space by a plan, just like the reprensetatin of space time made by this great physicist. We should also note that an application to the zeta function, would certainly yield interesting results?

However, this is not an exhaustive study. It is up to all the world's scientific community to improve and perfect it and each of them in there field of research.

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Appendices

A. Data for the development of table 6

To establish the data in table 6, we will use the imaginary product $\otimes^{i}_{e}$ defined in paragraph (4.4.1. p24)

- List of primary bases: $v_{1,i} = e^{\frac{\pi}{2}v_{1,i}}$; $v_{2,i} = e^{\frac{\pi}{2}v_{2,i}}$ and $v_{3,i} = e^{\frac{\pi}{2}v_{3,i}}$
- List of intermediate bases: $s_{1,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})}$; $s_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}$; $s_{3,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})}$; $s_{4,i} = e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})}$; $s_{5,i} = e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})}$ and $s_{6,i} = e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})}$
- List of upper bases: $u_{1,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})}$; $u_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}$; $u_{3,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})}$ and $u_{4,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})}$
- In the subset $\mathbb{S}_{(1;3)}$ we have: $v_{1,r} = v_{2,r} = v_{3,r} = v_r$ (where $v_{1,r} = e^{2k\pi v_{2,i}}$; $v_{2,r} = e^{2k\pi v_{3,i}}$; $v_{3,r} = e^{2k\pi v_{1,i}}$).

a. Multiplication of primary imaginary bases between them:

- $v_{1,i} \times v_{1,i} = (v_{1,i})^2 = \left(e^{\frac{\pi}{2}v_{1,i}}\right)^2 = e^{\pi v_{1,i}} = -e^{2k\pi v_{1,i}} = -v_{3,r} = -v_r$

In the following, we will use the same approach for calculating the squares of primary imaginary bases.

- $v_{1,i} \times v_{2,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{2,i}} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = s_{2,i}$
- $v_{2,i} \times v_{2,i} = (v_{2,i})^2 = \left(e^{\frac{\pi}{2}v_{2,i}}\right)^2 = e^{\pi v_{2,i}} = -v_r$
- $v_{2,i} \times v_{3,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}v_{3,i}} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = s_{3,i}$
- $v_{3,i} \times v_{3,i} = (v_{3,i})^2 = \left(e^{\frac{\pi}{2}v_{3,i}}\right)^2 = e^{\pi v_{3,i}} = -v_r$
- $v_{1,i} \times v_{3,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}v_{3,i}} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} = s_{1,i}$

b. Multiplication of the primary imaginary base $v_{1,i}$ with the intermediate and upper imaginary bases:

- $v_{1,i} \times s_{1,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} = e^{\frac{\pi}{2}(v_{3,i}+2v_{1,i})} = -e^{\frac{\pi}{2}v_{3,i}} = -v_{3,i}$
- $v_{1,i} \times s_{2,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = e^{\frac{\pi}{2}(2v_{1,i}+v_{2,i})} = -e^{\frac{\pi}{2}v_{2,i}} = -v_{2,i}$
- $v_{1,i} \times s_{3,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = u_{2,i}$
- $v_{1,i} \times s_{4,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(v_{3,i}+0v_{1,i})} = e^{\frac{\pi}{2}v_{3,i}} = v_{3,i}$
- $v_{1,i} \times s_{5,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(2v_{1,i}-v_{2,i})} = -e^{\frac{\pi}{2}(-v_{2,i})} = -(-v_{2,i}) = v_{2,i}$
- $v_{1,i} \times s_{6,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = u_{3,i}$
- $v_{1,i} \times u_{1,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+2v_{1,i}-v_{2,i})} = -e^{\frac{\pi}{2}(v_{3,i}-v_{2,i})} = -e^{-\frac{\pi}{2}(v_{2,i}-v_{3,i})} = -(-s_{6,i}) = s_{6,i}$
- $v_{1,i} \times u_{2,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+v_{2,i}+v_{3,i})} = -e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = -s_{3,i}$
- $v_{1,i} \times u_{3,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+v_{2,i}-v_{3,i})} = -e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} = -s_{6,i}$
- $v_{1,i} \times u_{4,i} = e^{\frac{\pi}{2}v_{1,i}} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}+0v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = s_{3,i}$

c. Multiplication of the primary imaginary base $v_{2,i}$ with the intermediate and upper imaginary bases:

- $v_{2,i} \times s_{1,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = u_{2,i}$
- $v_{2,i} \times s_{2,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i})} = -e^{\frac{\pi}{2}v_{1,i}} = -v_{1,i}$
- $v_{2,i} \times s_{3,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(2v_{2,i}+v_{3,i})} = -e^{\frac{\pi}{2}v_{3,i}} = -v_{3,i}$
- $v_{2,i} \times s_{4,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = u_{4,i}$
- $v_{2,i} \times s_{5,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{1,i}+0v_{2,i})} = e^{\frac{\pi}{2}v_{1,i}} = v_{1,i}$
- $v_{2,i} \times s_{6,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(2v_{2,i}-v_{3,i})} = -e^{\frac{\pi}{2}(-v_{3,i})} = -(-v_{3,i}) = v_{3,i}$
- $v_{2,i} \times u_{1,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}+0v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} = s_{1,i}$
- $v_{2,i} \times u_{2,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}+v_{3,i})} = -e^{\frac{\pi}{2}(v_{1,i}+v_{3,i})} = -s_{1,i}$

- $v_{2,i} \times u_{3,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}-v_{3,i})} = -e^{\frac{\pi}{2}(v_{1,i}-v_{3,i})} = -e^{-\frac{\pi}{2}(v_{3,i}-v_{1,i})} = -(-s_{4,i}) = s_{4,i}$
- $v_{2,i} \times u_{4,i} = e^{\frac{\pi}{2}v_{2,i}} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = -e^{\frac{\pi}{2}(2v_{2,i}+v_{3,i}-v_{1,i})} = -e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} = -s_{4,i}$

d. Multiplication of primary imaginary base $v_{3,i}$ with the intermediate imaginary and upper imaginary bases:

- $v_{3,i} \times s_{1,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} = e^{\frac{\pi}{2}(2v_{3,i}+v_{1,i})} = -e^{\frac{\pi}{2}v_{1,i}} = -v_{1,i}$
- $v_{3,i} \times s_{2,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = u_{2,i}$
- $v_{3,i} \times s_{3,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(v_{2,i}+2v_{3,i})} = -e^{\frac{\pi}{2}v_{2,i}} = -v_{2,i}$
- $v_{3,i} \times s_{4,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(2v_{3,i}-v_{1,i})} = -e^{-\frac{\pi}{2}v_{1,i}} = -(-v_{1,i}) = v_{1,i}$
- $v_{3,i} \times s_{5,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = u_{1,i}$
- $v_{3,i} \times s_{6,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{2,i}+0v_{3,i})} = e^{\frac{\pi}{2}v_{2,i}} = v_{2,i}$
- $v_{3,i} \times u_{1,i} = e^{\frac{\pi}{2}v_{3,i}} \times= e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(2v_{3,i}+v_{1,i}-v_{2,i})} = -e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} = -s_{5,i}$
- $v_{3,i} \times u_{2,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+2v_{3,i})} = -e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = -s_{2,i}$
- $v_{3,i} \times u_{3,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+0v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = s_{2,i}$
- $v_{3,i} \times u_{4,i} = e^{\frac{\pi}{2}v_{3,i}} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+2v_{3,i}-v_{1,i})} = -e^{\frac{\pi}{2}(v_{2,i}-v_{1,i})} = -e^{-\frac{\pi}{2}(v_{1,i}-v_{2,i})} = -(-s_{5,i}) = s_{5,i}$

e. Multiplication of intermediate imaginary bases between them:

- $s_{1,i} \times s_{1,i} = (s_{1,i})^2 = \left(e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})}\right)^2 = e^{\pi(v_{3,i}+v_{1,i})} = -v_r$

Here, it must be noted that:

$e^{\pi(v_{3,i}+v_{1,i})} = -e^{2k\pi(v_{3,i}+v_{1,i})} = -(e^{2k\pi v_{3,i}} \times e^{2k\pi v_{1,i}}) = -(v_{2,r} \times v_{3,r})$

Knowing that, in the subset $\mathbb{S}_{(1;3)}$ we had posed that; $v_{1,r} = v_{2,r} = v_{3,r} = v_r$

So $e^{\pi(v_{3,i}+v_{1,i})} = -(v_{2,r} \times v_{3,r}) = -(v_r)^2 = -v_r$

In the following, we will use the same approach for calculating the squares of intermediates or upper imaginary bases.

- $s_{1,i} \times s_{2,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = e^{\frac{\pi}{2}(2v_{1,i}+v_{2,i}+v_{3,i})} = -e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = -s_{3,i}$
- $s_{1,i} \times s_{3,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+2v_{3,i})} = -e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = -s_{2,i}$
- $s_{1,i} \times s_{4,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(2v_{3,i}+0v_{1,i})} = e^{\pi v_{3,i}} = -v_r$
- $s_{1,i} \times s_{5,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(2v_{1,i}+v_{3,i}-v_{2,i})} = -e^{\frac{\pi}{2}(v_{3,i}-v_{2,i})} = -e^{-\frac{\pi}{2}(v_{2,i}-v_{3,i})} = -(-s_{6,i}) = s_{6,i}$
- $s_{1,i} \times s_{6,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+0v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} = s_{2,i}$
- $s_{2,i} \times s_{2,i} = (s_{2,i})^2 = \left(e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})}\right)^2 = e^{\pi(v_{1,i}+v_{2,i})} = -v_r$
- $s_{2,i} \times s_{3,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}+v_{3,i})} = -e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} = -s_{1,i}$
- $s_{2,i} \times s_{4,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}+0v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} = s_{3,i}$
- $s_{2,i} \times s_{5,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(2v_{1,i}+0v_{2,i})} = e^{\pi v_{1,i}} = -v_r$
- $s_{2,i} \times s_{6,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}-v_{3,i})} = -e^{\frac{\pi}{2}(v_{1,i}-v_{3,i})} = -e^{-\frac{\pi}{2}(v_{3,i}-v_{1,i})} = -(-s_{4,i}) = s_{4,i}$
- $s_{3,i} \times s_{3,i} = (s_{3,i})^2 = \left(e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})}\right)^2 = e^{\pi(v_{2,i}+v_{3,i})} = -v_r$
- $s_{3,i} \times s_{4,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+2v_{3,i}-v_{1,i})} = -e^{\frac{\pi}{2}(v_{2,i}-v_{1,i})} = -e^{-\frac{\pi}{2}(v_{1,i}-v_{2,i})} = -(-s_{5,i}) = s_{5,i}$
- $s_{3,i} \times s_{5,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}+0v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} = s_{1,i}$
- $s_{3,i} \times s_{6,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(2v_{2,i}+0v_{3,i})} = e^{\pi v_{2,i}} = -v_r$
- $s_{4,i} \times s_{4,i} = (s_{4,i})^2 = \left(e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})}\right)^2 = e^{\pi(v_{3,i}-v_{1,i})} = -v_r$
- $s_{4,i} \times s_{5,i} = e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} \times e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+0v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}-v_{2,i})} = e^{-\frac{\pi}{2}(v_{2,i}-v_{3,i})} = -s_{6,i}$
- $s_{4,i} \times s_{6,i} = e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} \times e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{2,i}-v_{1,i}+0v_{3,i})} = e^{\frac{\pi}{2}(v_{2,i}-v_{1,i})} = e^{-\frac{\pi}{2}(v_{1,i}-v_{2,i})} = -s_{5,i}$

- $s_{5,i} \times s_{5,i} = (s_{5,i})^2 = \left(e^{\frac{\pi}{2}(v_{1,i} - v_{2,i})}\right)^2 = e^{\pi(v_{1,i} - v_{2,i})} = -v_r$
- $s_{5,i} \times s_{6,i} = e^{\frac{\pi}{2}(v_{1,i} - v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i} - v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i} + 0v_{2,i} - v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i} - v_{3,i})} = e^{-\frac{\pi}{2}(v_{3,i} - v_{1,i})} = -s_{4,i}$
- $s_{6,i} \times s_{6,i} = (s_{6,i})^2 = \left(e^{\frac{\pi}{2}(v_{2,i} - v_{3,i})}\right)^2 = e^{\pi(v_{2,i} - v_{3,i})} = -v_r$

f. Multiplication of the intermediate imaginary base $s_{1,i}$ with the upper imaginary bases:

- $s_{1,i} \times u_{1,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(2v_{3,i}+2v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}[2(v_{3,i}+v_{1,i})-v_{2,i}]} = -e^{-\frac{\pi}{2}v_{2,i}} = -(-v_{2,i}) = v_{2,i}$

  Note: here, in the imaginary product, we give priority to the intermediate imaginary base $s_{1,i} = v_{3,i} + v_{1,i}$.

  Which gives us: $e^{\frac{\pi}{2}[2(v_{3,i}+v_{1,i})-v_{2,i}]} = e^{\frac{\pi}{2}[2(s_{1,i})-v_{2,i}]} = -e^{-\frac{\pi}{2}v_{2,i}}$

  So in similar cases, priority is given to intermediate or upper bases.
- $s_{1,i} \times u_{2,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+v_{2,i}+2v_{3,i})} = e^{\frac{\pi}{2}[2(v_{1,i}+v_{3,i})+v_{2,i}]} = -e^{\frac{\pi}{2}v_{2,i}} = -v_{2,i}$
- $s_{1,i} \times u_{3,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+v_{2,i}+0v_{3,i})} = -e^{\frac{\pi}{2}v_{2,i}} = -v_{2,i}$
- $s_{1,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+2v_{3,i}+0v_{1,i})} = -e^{\frac{\pi}{2}v_{2,i}} = -v_{2,i}$

g. Multiplication of the intermediate imaginary base $s_{2,i}$ with the upper imaginary bases:

- $s_{2,i} \times u_{1,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+2v_{1,i}+0v_{2,i})} = -e^{\frac{\pi}{2}v_{3,i}} = -v_{3,i}$
- $s_{2,i} \times u_{2,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+2v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}[2(v_{1,i}+v_{2,i})+v_{3,i}]} = -e^{\frac{\pi}{2}v_{3,i}} = -v_{3,i}$
- $s_{2,i} \times u_{3,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+2v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}[2(v_{1,i}+v_{2,i})-v_{3,i}]} = -e^{-\frac{\pi}{2}v_{3,i}} = -(-v_{3,i}) = v_{3,i}$
- $s_{2,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(2v_{2,i}+v_{3,i}+0v_{1,i})} = -e^{\frac{\pi}{2}v_{3,i}} = -v_{3,i}$

h. Multiplication of the intermediate imaginary base $s_{3,i}$ with the upper imaginary bases:

- $s_{3,i} \times u_{1,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(2v_{3,i}+v_{1,i}+0v_{2,i})} = -e^{\frac{\pi}{2}v_{1,i}} = -v_{1,i}$
- $s_{3,i} \times u_{2,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}+2v_{3,i})} = e^{\frac{\pi}{2}[v_{1,i}+2(v_{2,i}+v_{3,i})]} = -e^{\frac{\pi}{2}v_{1,i}} = -v_{1,i}$
- $s_{3,i} \times u_{3,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}+0v_{3,i})} = -e^{\frac{\pi}{2}v_{1,i}} = -v_{1,i}$
- $s_{3,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(2v_{2,i}+2v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}[2(v_{2,i}+v_{3,i})-v_{1,i}]} = -e^{-\frac{\pi}{2}v_{1,i}} = -(-v_{1,i}) = v_{1,i}$

i. Multiplication of the intermediate imaginary base $s_{4,i}$ with the upper imaginary bases:

- $s_{4,i} \times u_{1,i} = e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(2v_{3,i}+0v_{1,i}-v_{2,i})} = -e^{-\frac{\pi}{2}v_{2,i}} = -(-v_{2,i}) = v_{2,i}$
- $s_{4,i} \times u_{2,i} = e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(0v_{1,i}+v_{2,i}+2v_{3,i})} = -e^{\frac{\pi}{2}v_{2,i}} = -v_{2,i}$
- $s_{4,i} \times u_{3,i} = e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(0v_{1,i}+v_{2,i}+0v_{3,i})} = e^{\frac{\pi}{2}v_{2,i}} = v_{2,i}$
- $s_{4,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{3,i}-v_{1,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(v_{2,i}+2v_{3,i}-2v_{1,i})} = e^{\frac{\pi}{2}[v_{2,i}+2(v_{3,i}-v_{1,i})]} = -e^{\frac{\pi}{2}v_{2,i}} = -v_{2,i}$

j. Multiplication of the intermediate imaginary base $s_{5,i}$ with the upper imaginary bases:

- $s_{5,i} \times u_{1,i} = e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(v_{3,i}+2v_{1,i}-2v_{2,i})} = e^{\frac{\pi}{2}[v_{3,i}+2(v_{1,i}-v_{2,i})]} = -e^{\frac{\pi}{2}v_{3,i}} = -v_{3,i}$
- $s_{5,i} \times u_{2,i} = e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+0v_{2,i}+v_{3,i})} = -e^{\frac{\pi}{2}v_{3,i}} = -v_{3,i}$
- $s_{5,i} \times u_{3,i} = e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+0v_{2,i}-v_{3,i})} = -e^{-\frac{\pi}{2}v_{3,i}} = -(-v_{3,i}) = v_{3,i}$
- $s_{5,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{1,i}-v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(0v_{2,i}+v_{3,i}+0v_{1,i})} = e^{\frac{\pi}{2}v_{3,i}} = v_{3,i}$

k. Multiplication of the intermediate imaginary base $s_{6,i}$ with the upper imaginary bases:

- $s_{6,i} \times u_{1,i} = e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} \times e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} = e^{\frac{\pi}{2}(0v_{3,i}+v_{1,i}+0v_{2,i})} = e^{\frac{\pi}{2}v_{1,i}} = v_{1,i}$
- $s_{6,i} \times u_{2,i} = e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}+0v_{3,i})} = -e^{\frac{\pi}{2}v_{1,i}} = -v_{1,i}$
- $s_{6,i} \times u_{3,i} = e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(v_{1,i}+2v_{2,i}-2v_{3,i})} = e^{\frac{\pi}{2}[v_{1,i}+2(v_{2,i}-v_{3,i})]} = -e^{\frac{\pi}{2}v_{1,i}} = -v_{1,i}$
- $s_{6,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{2,i}-v_{3,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(2v_{2,i}+0v_{3,i}-v_{1,i})} = -e^{-\frac{\pi}{2}v_{1,i}} = -(-v_{1,i}) = v_{1,i}$

l. Multiplication of upper imaginary bases between them:

- $u_{1,i} \times u_{1,i} = (u_{1,i})^2 = \left(e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})}\right)^2 = e^{\pi(v_{3,i}+v_{1,i}-v_{2,i})} = -v_r$

- $u_{1,i} \times u_{2,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+0v_{2,i}+2v_{3,i})} = e^{\frac{\pi}{2}[2(v_{1,i}+v_{3,i})]} = e^{\pi(v_{1,i}+v_{3,i})} = -v_r$
- $u_{1,i} \times u_{3,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+0v_{2,i}+0v_{3,i})} = e^{\pi v_{1,i}} = -v_r$
- $u_{1,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{3,i}+v_{1,i}-v_{2,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(0v_{2,i}+2v_{3,i}+0v_{1,i})} = e^{\pi v_{3,i}} = -v_r$
- $u_{2,i} \times u_{2,i} = (u_{2,i})^2 = \left(e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})}\right)^2 = e^{\pi(v_{1,i}+v_{2,i}+v_{3,i})} = -v_r$
- $u_{2,i} \times u_{3,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} = e^{\frac{\pi}{2}(2v_{1,i}+2v_{2,i}+0v_{3,i})} = e^{\frac{\pi}{2}[2(v_{1,i}+v_{2,i})]} = e^{\pi(v_{1,i}+v_{2,i})} = -v_r$
- $u_{2,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}+v_{3,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(2v_{2,i}+2v_{3,i}+0v_{1,i})} = e^{\frac{\pi}{2}[2(v_{2,i}+v_{3,i})]} = e^{\pi(v_{2,i}+v_{3,i})} = -v_r$
- $u_{3,i} \times u_{3,i} = (u_{3,i})^2 = \left(e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})}\right)^2 = e^{\pi(v_{1,i}+v_{2,i}-v_{3,i})} = -v_r$
- $u_{3,i} \times u_{4,i} = e^{\frac{\pi}{2}(v_{1,i}+v_{2,i}-v_{3,i})} \times e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})} = e^{\frac{\pi}{2}(2v_{2,i}+0v_{3,i}+0v_{1,i})} = e^{\pi v_{2,i}} = -v_r$
- $u_{4,i} \times u_{4,i} = (u_{4,i})^2 = \left(e^{\frac{\pi}{2}(v_{2,i}+v_{3,i}-v_{1,i})}\right)^2 = e^{\pi(v_{2,i}+v_{3,i}-v_{1,i})} = -v_r$

We therefore obtain the commutative table for the subset $\mathbb{S}_{(1;3)}$

$\times$	v_r	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$	$s_{1,i}$	$s_{2,i}$	$s_{3,i}$	$s_{4,i}$	$s_{5,i}$	$s_{6,i}$	$u_{1,i}$	$u_{2,i}$	$u_{3,i}$	$u_{4,i}$
v_r	v_r	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$	$s_{1,i}$	$s_{2,i}$	$s_{3,i}$	$s_{4,i}$	$s_{5,i}$	$s_{6,i}$	$u_{1,i}$	$u_{2,i}$	$u_{3,i}$	$u_{4,i}$
$v_{1,i}$	$v_{1,i}$	$-v_r$	$s_{2,i}$	$s_{1,i}$	$-v_{3,i}$	$-v_{2,i}$	$u_{2,i}$	$v_{3,i}$	$v_{2,i}$	$u_{3,i}$	$s_{6,i}$	$-s_{3,i}$	$-s_{6,i}$	$s_{3,i}$
$v_{2,i}$	$v_{2,i}$	$s_{2,i}$	$-v_r$	$s_{3,i}$	$u_{2,i}$	$-v_{1,i}$	$-v_{3,i}$	$u_{4,i}$	$v_{1,i}$	$v_{3,i}$	$s_{1,i}$	$-s_{1,i}$	$s_{4,i}$	$-s_{4,i}$
$v_{3,i}$	$v_{3,i}$	$s_{1,i}$	$s_{3,i}$	$-v_r$	$-v_{1,i}$	$u_{2,i}$	$-v_{2,i}$	$v_{1,i}$	$u_{1,i}$	$v_{2,i}$	$-s_{5,i}$	$-s_{2,i}$	$s_{2,i}$	$s_{5,i}$
$s_{1,i}$	$s_{1,i}$	$-v_{3,i}$	$u_{2,i}$	$-v_{1,i}$	$-v_r$	$-s_{3,i}$	$-s_{2,i}$	$-v_r$	$s_{6,i}$	$s_{2,i}$	$v_{2,i}$	$-v_{2,i}$	$-v_{2,i}$	$-v_{2,i}$
$s_{2,i}$	$s_{2,i}$	$-v_{2,i}$	$-v_{1,i}$	$u_{2,i}$	$-s_{3,i}$	$-v_r$	$-s_{1,r}$	$s_{3,i}$	$-v_r$	$s_{4,i}$	$-v_{3,i}$	$-v_{3,i}$	$v_{3,i}$	$-v_{3,i}$
$s_{3,i}$	$s_{3,i}$	$u_{2,i}$	$-v_{3,i}$	$-v_{2,i}$	$-s_{2,i}$	$-s_{1,r}$	$-v_r$	$s_{5,i}$	$s_{1,i}$	$-v_r$	$-v_{1,i}$	$-v_{1,i}$	$-v_{1,i}$	$v_{1,i}$
$s_{4,i}$	$s_{4,i}$	$v_{3,i}$	$u_{4,i}$	$v_{1,i}$	$-v_r$	$s_{3,i}$	$s_{5,i}$	$-v_r$	$-s_{6,i}$	$-s_{5,i}$	$v_{2,i}$	$-v_{2,i}$	$v_{2,i}$	$-v_{2,i}$
$s_{5,i}$	$s_{5,i}$	$v_{2,i}$	$v_{1,i}$	$u_{1,i}$	$s_{6,i}$	$-v_r$	$s_{1,i}$	$-s_{6,i}$	$-v_r$	$-s_{4,i}$	$-v_{3,i}$	$-v_{3,i}$	$v_{3,i}$	$v_{3,i}$
$s_{6,i}$	$s_{6,i}$	$u_{3,i}$	$v_{3,i}$	$v_{2,i}$	$s_{2,i}$	$s_{4,i}$	$-v_r$	$-s_{5,i}$	$-s_{4,i}$	$-v_r$	$v_{1,i}$	$-v_{1,i}$	$-v_{1,i}$	$v_{1,i}$
$u_{1,i}$	$u_{1,i}$	$s_{6,i}$	$s_{1,i}$	$-s_{5,i}$	$v_{2,i}$	$-v_{3,i}$	$-v_{1,i}$	$v_{2,i}$	$-v_{3,i}$	$v_{1,i}$	$-v_r$	$-v_r$	$-v_r$	$-v_r$
$u_{2,i}$	$u_{2,i}$	$-s_{3,i}$	$-s_{1,i}$	$-s_{2,i}$	$-v_{2,i}$	$-v_{3,i}$	$-v_{1,i}$	$-v_{2,i}$	$-v_{3,i}$	$-v_{1,i}$	$-v_r$	$-v_r$	$-v_r$	$-v_r$
$u_{3,i}$	$u_{3,i}$	$-s_{6,i}$	$s_{4,i}$	$s_{2,i}$	$-v_{2,i}$	$v_{3,i}$	$-v_{1,i}$	$v_{2,i}$	$v_{3,i}$	$-v_{1,i}$	$-v_r$	$-v_r$	$-v_r$	$-v_r$
$u_{4,i}$	$u_{4,i}$	$s_{3,i}$	$-s_{4,i}$	$s_{5,i}$	$-v_{2,i}$	$-v_{3,i}$	$v_{1,i}$	$v_{2,i}$	$v_{3,i}$	$v_{1,i}$	$-v_r$	$-v_r$	$-v_r$	$-v_r$

B. Data for the development of table 7

- List of primary bases: $i = e^{\frac{\pi}{2}i}$ and $j = e^{\frac{\pi}{2}j}$
- List of upper bases: $e^{\frac{\pi}{2}(i+j)}$ and $e^{\frac{\pi}{2}(i-j)}$

a. Multiplication of primary imaginary bases between them:

- $e^{\frac{\pi}{2}i} \times e^{\frac{\pi}{2}i} = \left(e^{\frac{\pi}{2}i}\right)^2 = e^{\pi i} = -e^{2k\pi i} = -1$
- $e^{\frac{\pi}{2}i} \times e^{\frac{\pi}{2}j} = e^{\frac{\pi}{2}(i+j)}$
- $e^{\frac{\pi}{2}j} \times e^{\frac{\pi}{2}j} = \left(e^{\frac{\pi}{2}j}\right)^2 = e^{\pi j} = -e^{2k\pi j} = -1$

b. Multiplication of primary imaginary base $e^{\frac{\pi}{2}i}$ with the upper imaginary bases:

- $e^{\frac{\pi}{2}i} \times e^{\frac{\pi}{2}(i+j)} = e^{\frac{\pi}{2}(2i+j)} = -e^{\frac{\pi}{2}j}$
- $e^{\frac{\pi}{2}i} \times e^{\frac{\pi}{2}(i-j)} = e^{\frac{\pi}{2}(2i-j)} = -e^{-\frac{\pi}{2}j} = e^{\frac{\pi}{2}j}$

c. Multiplication of primary imaginary base $e^{\frac{\pi}{2}j}$ with the upper imaginary bases:

- $e^{\frac{\pi}{2}j} \times e^{\frac{\pi}{2}(i+j)} = e^{\frac{\pi}{2}(i+2j)} = -e^{\frac{\pi}{2}i}$
- $e^{\frac{\pi}{2}j} \times e^{\frac{\pi}{2}(i-j)} = e^{\frac{\pi}{2}(i+0j)} = e^{\frac{\pi}{2}i}$

d. Multiplication of upper imaginary bases between them:

- $e^{\frac{\pi}{2}(i+j)} \times e^{\frac{\pi}{2}(i+j)} = \left(e^{\frac{\pi}{2}(i+j)}\right)^2 = e^{\pi(i+j)} = -e^{2k\pi(i+j)} = -\left(e^{2k\pi i} \times e^{2k\pi j}\right) = -(1 \times 1) = -1$
- $e^{\frac{\pi}{2}(i+j)} \times e^{\frac{\pi}{2}(i-j)} = e^{\frac{\pi}{2}(2i+0j)} = e^{\pi i} = -e^{2k\pi i} = -1$
- $e^{\frac{\pi}{2}(i-j)} \times e^{\frac{\pi}{2}(i-j)} = \left(e^{\frac{\pi}{2}(i-j)}\right)^2 = \left(e^{\frac{\pi}{2}(i-j)}\right)^2 = e^{\pi(i-j)} = -e^{2k\pi(i-j)} = -\left(e^{2k\pi i} \times e^{-2k\pi j}\right) = -(1 \times 1) = -1$

We must remember that: $e^{-\pi j} = e^{\pi j} = -1$ and $e^{-2k\pi j} = e^{2k\pi j} = 1$

We thus obtain the commutative table:

$\times$	1	$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i-j)}$
1	1	$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i-j)}$
$e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}i}$	-1	$e^{\frac{\pi}{2}(i+j)}$	$-e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}j}$
$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}(i+j)}$	-1	$-e^{\frac{\pi}{2}i}$	$e^{\frac{\pi}{2}i}$
$e^{\frac{\pi}{2}(i+j)}$	$e^{\frac{\pi}{2}(i+j)}$	$-e^{\frac{\pi}{2}j}$	$-e^{\frac{\pi}{2}i}$	-1	-1
$e^{\frac{\pi}{2}(i-j)}$	$e^{\frac{\pi}{2}(i-j)}$	$e^{\frac{\pi}{2}j}$	$e^{\frac{\pi}{2}i}$	-1	-1