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Fixed Point Theorems on a generalized N_b-metric space

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Abstract:
In this paper, we introduce a notion of a generalized N_b-metric space, which is a generalization of N-metric spaces. This new space considers the path integral in physics. The inspiration comes from the expression of quantum mechanical amplitude for a particle to go from the initial point x to the final point y

$$\int Dqe^{(i/h)S(q)}.$$

Fixed point theorems satisfying some contractive conditions are stated and proved. This concept generalizes some known results in the literature.

Keywords: N-metric space, N_b-metric space, Cauchy sequence

1 Introduction

Banach fixed point theorem was proved in 1922. This theorem gave conditions for the existence of a unique fixed point for a self-map defined on a metric space with completeness. After this result, much literature sprang up for continuous maps until Kannan proved the existence of a fixed point for contractive maps that does not imply continuity. Motivated by these results, Branciari and many other authors tried to prove the existence of fixed points for contractive maps in other spaces. (see [1-34]).
In this paper, we introduce the notion of generalized N_b-metric spaces, which is a generalization of N-metric spaces. This new space considers the path integral in physics. The inspiration comes from the expression of quantum mechanical amplitude for a particle to go from the initial point x to the final point y

$$\int Dqe^{(i/h)S(q)}.$$

Fixed point theorems satisfying Banach contractive condition, Kanaan contraction, Chatterjea Type contraction, Zamfirescu's contraction, and general contractive condition of integral type are stated and proved.

2 Main results

We introduce N_b-metric space inspired by the path integral in physics. We define as follows.
Definition 2.1.
For a non-empty set X and a function $d_b : X^2 \to [0, \infty)$ satisfying the following properties:

N_1 $d(x_i, x_j, N_b) = 0 \iff i = j$.

N_2 $d(x_i, x_j, N_b) = d(x_j, x_i, N_b)$.

N_3 $d(x_i, x_N, N_b) \le b\left[\sum_{i=1}^{N-1} d(x_i, x_{i+1}, N_b)\right], b \ge 1$.

for all $x_i, x_j \in X$ and $i, j = 0, 1, 2, ..., N, i \ne j$, (X, d_b) is called a generalized N_b-metric space.

The above definition draws its inspiration from the expression of quantum mechanical amplitude for a particle to go from the initial point x to the final point y:

$$\int Dqe^{(i/h)S(q)}. \tag{1}$$

where q is the position of the particle, $\int Dq$ the sum of all possible paths between x and y,

$$S(q) = \int_0^T dtL(q, \overline{q}) \tag{2}$$

the classical action, h the Planck's constant. See [11, 17-21]. The result for quantum mechanics is that the classical path between the two points has the largest weights and quantum effects give fluctuations around it.
The approach used is to modify the measure weight in (1) by taking the simple case of $S(q) = 0$ and proceeding with Definition 2.1. This generalization extends and improves the idea of [9] and many other results in literature (see [1-34]).
Example 2.2.
Let $X = P \cup \mathbb{Z}$, $P = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ and define $d_b : X^2 \to R^+ \cup \{0\}$ by

$$d_b(x, y, N_b) = \begin{cases} 0, & x = y; \\ |x|, & x, y \in P; \\ |y|, & otherwise \end{cases}$$

Then (X, d_b, N_b) is a generalized N_b-metric space but not necessarily a metric space nor N-metric space because
$d_b\left(\frac{1}{2}, \frac{1}{32}, N_b\right) \ge d_b\left(\frac{1}{2}, \frac{2}{55}, N_b\right) + d_b\left(\frac{2}{55}, \frac{3}{55}, N_b\right) + d_b\left(\frac{3}{55}, \frac{1}{32}, N_b\right)$
$\implies \frac{1}{2} > \frac{2}{55} + \frac{3}{55} + \frac{1}{32}$.
For $s = 5, 6, 7, ...,$
$d_b\left(\frac{1}{2}, \frac{1}{32}, N_b\right) \le s[d_b\left(\frac{1}{2}, \frac{2}{55}, N_b\right) + d_b\left(\frac{2}{55}, \frac{3}{55}, N_b\right) + d_b\left(\frac{3}{55}, \frac{1}{32}, N_b\right)]$

Example 2.3.
Let $X = \mathbb{R}$ and define $d_b : X^2 \to R^+ \cup \{0\}$ by

$$d_b(x,y,N_b) = \begin{cases} 0, & x = y; \\ |x|, & x \in \mathbb{Q}; \\ y^2, & otherwise \end{cases}$$

Then (X, d_b, N_b) is a generalized N_b-metric space.
Example 2.4.
Let $X = \mathbb{R}$ and define $d_b : X^2 \to R^+ \cup \{0\}$ by

$$d_b(x,y,N_b) = \begin{cases} 0, & x = y; \\ \frac{1}{2}, & x, y \in \mathbb{R} \end{cases}$$

Then (X, d_b, N_b) is a generalized N_b-metric space.

Definition 2.5. Let (X, d_b) be a generalized N_b-metric space. For $y \in X$, $r > 0$, the d_b-sphere with centre y and radius r is

$$S_{d_b}(y,r) = \{z \in X : d_b(y,z,N_b) < r\}$$

Definition 2.6. Let (X, d_b) be a generalized N_b-metric space. A sequence $\{x_n\} \subset X$ is d_b-convergent to z if the limit of $d_b(x_n, z, N_b)$ tends to zero as n tends to infinity.

Definition 2.7. Let (X, d_b) and $(\overline{X}, \overline{d_b})$ be two generalized N_b-metric spaces, a function $g : X \to \overline{X}$ is d_b-continuous at a point $x \in X$ if $g^{-1}(S_{\overline{d_b}}(g(x), r)) \in \overline{X}$, for all $r > 0$. g is d_b-continuous if it is d_b-continuous at all points of X.

Lemma 2.8. (X, d_b) be a generalized N_b-metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to x_o if and only if $d_b(x_n, x_o, N_b) \to 0$ as $n \to \infty$.

Lemma 2.9. (X, d_b) be a generalized N_b-metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be a Cauchy sequence if and only if $d_b(x_i, x_N, N_b) \to 0$ as $i, N \to \infty$.

Theorem 2.10. Let X be a complete N_b-metric space and $g : X \to X$ a map for which there exist the real number, q satisfying $0 \le q < 1$ such that for each pair $x, y \in X$.

$$d_b(gx, gy, N_b) \le q d_b(x, y, N_b) \tag{3}$$

Then g has a unique fixed point.
Proof:
Suppose g satisfies condition (3) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = g^n x_0$, then
$d_b(x_n, x_{n+1}, N_b) = d_b(gx_{n-1}, gx_n, N_b) \le q d_b(x_{n-1}, x_n, N_b)$
Setting $h_n = d_b(x_n, x_{n+1}, N_b)$ we have

$$h_n \le q h_{n-1} \tag{4}$$

We then deduce that

$$\begin{aligned} h_n &\leq qh_{n-1} && (5) \\ h_n &\leq q^n h_0 \forall n \in N. && (6) \end{aligned}$$

Using (N_3) of Definition 2.1, we obtain

$$\begin{aligned} d_b(x_i, x_N, N_b) &\leq b\left[\sum_{i=1}^{N-1} d(x_i, x_{i+1}, N_b)\right] && (7) \\ &= b\left[\sum_{i=1}^{N-1} h_i\right] && (8) \\ &= b\left[\sum_{i=1}^{N-1} q^i h_0\right] && (9) \end{aligned}$$

Taking the limit of $d_b(x_i, x_N, N_b)$ as $N, i \to \infty$, we have

$$\lim_{n,m\to\infty} d_b(x_i, x_N, N_b) = \lim_{n,m\to\infty} b\left[\sum_{i=1}^{N-1} q^i h_0\right] = 0 \quad (10)$$

So, $\{x_n\}$ is a d_b-Cauchy Sequence.
By completeness of (X, d_b), there exist $x_o \in X$ such that x_n is d_b-convergent to x_o.
Suppose $gx_o \neq x_o$

$$d_b(x_n, gx_o, N_b) \leq qd_b(x_{n-1}, x_o, N_b). \quad (11)$$

Taking the limit as $n \to \infty$ and using the fact that function is d_b-continuous in its variables, we get

$$d_b(x_o, gx_o, N_b) \leq qd_b(x_o, x_o, N_b). \quad (12)$$

Hence,

$$d_b(x_o, gx_o, N_b) \leq 0. \quad (13)$$

This is a contradiction. So, $gx_o = x_o$.
To show the uniqueness, suppose $x_1 \neq x_2$ is such that $gx_1 = x_1$ and $gx_2 = x_2$ then

$$d_b(gx_1, gx_2, N_b) \leq qd_b(x_1, x_2, N_b). \quad (14)$$

Since $gx_1 = x_1$ and $gx_2 = x_2$, we have

$$d_b(x_1, x_2, N_b) \leq 0. \quad (15)$$

which implies that $x_1 = x_2$.

Remark 2.11. Let (X, d_b) be a generalized N_b-metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = d_b(x, y, N_b)$, then Theorem 2.10 reduces to Banach contraction principle in a generalized N-metric space(an analogue of Banach contraction principle in metric space).

Theorem 2.12. Let X be a complete N_b-metric space and $g : X \to X$ a map for which there exist the real number, c satisfying $0 \leq c < \frac{1}{2}$ such that for each pair $x, y \in X$.

$$d_b(gx, gy, N_b) \leq c[d_b(x, gx, N_b) + d_b(y, gy, N_b)] \quad (16)$$

Then g has a unique fixed point.
Proof:
Suppose g satisfies condition (16) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = g^n x_0$, then

$$\begin{aligned} d_b(x_n, x_{n+1}, N_b) &= d_b(gx_{n-1}, gx_n, N_b) && (17)\\ &\leq c[d_b(x_{n-1}, gx_{n-1}, N_b) + d_b(x_n, gx_n, N_b)] && (18)\\ &\leq \left[\frac{c}{1-c}\right] d_b(x_n, x_{n-1}, N_b) && (19) \end{aligned}$$

Setting $h_n = d_b(x_n, x_{n+1}, N_b)$ and $q = \left[\frac{c}{1-c}\right]$, we have

$$h_n \leq q h_{n-1} \quad (20)$$

We then deduce that

$$\begin{aligned} h_n &\leq q h_{n-1} && (21)\\ h_n &\leq q^n h_0 \forall n \in N. && (22) \end{aligned}$$

Using (N_3) of Definition 2.1, we obtain

$$\begin{aligned} d_b(x_i, x_N, N_b) &\leq b\left[\sum_{i=1}^{N-1} d(x_i, x_{i+1}, N_b)\right] && (23)\\ &= b\left[\sum_{i=1}^{N-1} h_i\right] && (24)\\ &= b\left[\sum_{i=1}^{N-1} q^i h_0\right] && (25) \end{aligned}$$

Taking the limit of $d_b(x_i, x_N, N_b)$ as $N, i \to \infty$, we have

$$\lim_{n,m\to\infty} d_b(x_i, x_N, N_b) = \lim_{n,m\to\infty} b\left[\sum_{i=1}^{N-1} q^i h_0\right] = 0 \quad (26)$$

So, $\{x_n\}$ is a d_b-Cauchy Sequence.
By completeness of (X, d_b), there exist $x_o \in X$ such that x_n is d_b-convergent to x_o.
Suppose $gx_o \neq x_o$

$$d_b(x_n, gx_o, N_b) \leq c[d_b(gx_{n-1}, x_{n-1}, N_b) + d_b(gx_o, x_o, N_b)]. \quad (27)$$

Taking the limit as $n \to \infty$ and using the fact that function is d_b-continuous in its variables, we get

$$d_b(x_o, gx_o, N_b) \leq c[d_b(gx_o, x_o, N_b) + d_b(gx_o, x_o, N_b)]. \quad (28)$$

Hence,

$$d_b(x_o, gx_o, N_b) \leq 0. \tag{29}$$

This is a contradiction. So, $gx_o = x_o$.
To show the uniqueness, suppose $x_1 \neq x_2$ is such that $gx_1 = x_1$ and $gx_2 = x_2$ then

$$d_b(gx_1, gx_2, N_b) \leq c[d_b(gx_1, x_1, N_b) + d_b(gx_2, x_2, N_b)]. \tag{30}$$

Since $gx_1 = x_1$ and $gx_2 = x_2$, we have

$$d_b(x_1, x_2, N_b) \leq 0. \tag{31}$$

which implies that $x_1 = x_2$.

Remark 2.13. Let (X, d_b) be a generalized N_b-metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = d_b(x, y)$, then Theorem 2.12 reduces to Kanaan contraction in a generalized N-metric space(an analogue of Kanaan contraction in metric space).

Theorem 2.14. Let X be a complete N_b-metric space and $g : X \to X$ a map for which there exist the real number, c satisfying $0 \leq c < \frac{1}{2}$ and $c < \frac{1}{b+1}$ such that for each pair $x, y \in X$.

$$d_b(gx, gy, N_b) \leq c[d_b(x, gy, N_b) + d_b(y, gx, N_b)] \tag{32}$$

Then g has a unique fixed point.
Proof:
Suppose g satisfies condition (32) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = g^n x_0$, then

$$\begin{aligned}
d_b(x_n, x_{n+1}, N_b) &= d_b(gx_{n-1}, gx_n, N_b) && (33)\\
&\leq c[d_b(gx_{n-1}, x_n, N_b) + d_b(gx_n, x_{n-1}, N_b)] && (34)\\
&\leq c[d_b(x_n, x_n, N_b) + d_b(x_{n+1}, x_{n-1}, N_b)] && (35)\\
&\leq bc[d_b(x_{n+1}, x_n, N_b) + d_b(x_n, x_{n-1}, N_b)] && (36)\\
&\leq \left[\frac{c}{1 - bc}\right] d_b(x_n, x_{n-1}, N_b) && (37)
\end{aligned}$$

Setting $h_n = d_b(x_n, x_{n+1}, N_b)$ and $q = \left[\frac{c}{1-bc}\right]$, we have

$$h_n \leq qh_{n-1} \tag{38}$$

We then deduce that

$$\begin{aligned}
h_n &\leq qh_{n-1} && (39)\\
h_n &\leq q^n h_0 \forall n \in N. && (40)
\end{aligned}$$

Using (N_3) of Definition 2.1, we obtain

$$\begin{aligned} d_b(x_i, x_N, N_b) &\leq b\left[\sum_{i=1}^{N-1} d(x_i, x_{i+1}, N_b)\right] &(41)\\ &= b\left[\sum_{i=1}^{N-1} h_i\right] &(42)\\ &= b\left[\sum_{i=1}^{N-1} q^i h_0\right] &(43)\end{aligned}$$

Taking the limit of $d_b(x_i, x_N, N_b)$ as $N, i \to \infty$, we have

$$\lim_{n,m\to\infty} d_b(x_i, x_N, N_b) = \lim_{n,m\to\infty} b\left[\sum_{i=1}^{N-1} q^i h_0\right] = 0 \quad (44)$$

So, $\{x_n\}$ is a d_b-Cauchy Sequence.
By completeness of (X, d_b), there exist $x_o \in X$ such that x_n is d_b-convergent to x_o.
Suppose $gx_o \neq x_o$

$$d_b(x_n, gx_o, N_b) \leq c[d_b(gx_{n-1}, x_o, N_b) + d_b(gx_o, x_{n-1}, N_b)]. \quad (45)$$

Taking the limit as $n \to \infty$ and using the fact that function is d_b-continuous in its variables, we get

$$d_b(x_o, gx_o, N_b) \leq c[d_b(gx_o, x_o, N_b) + d_b(gx_o, x_o, N_b)]. \quad (46)$$

Hence,

$$d_b(x_o, gx_o, N_b) \leq 0. \quad (47)$$

This is a contradiction. So, $gx_o = x_o$.
To show the uniqueness, suppose $x_1 \neq x_2$ is such that $gx_1 = x_1$ and $gx_2 = x_2$ then

$$d_b(gx_1, gx_2, N_b) \leq c[d_b(gx_1, x_2, N_b) + d_b(gx_2, x_1, N_b)]. \quad (48)$$

Since $gx_1 = x_1$ and $gx_2 = x_2$, we have

$$d_b(x_1, x_2, N_b) \leq 0. \quad (49)$$

which implies that $x_1 = x_2$.

Remark 2.15. Let (X, d_b) be a generalized N_b-metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = d_b(x, y)$, then Theorem 2.14 reduces to Chatterjea Type Contraction in a generalized N-metric space(an analogue of Chatterjea Type Contraction in metric space).

Theorem 2.16. Let X be a complete N_b-metric space and $g : X \to X$ a map for which there exists real numbers, $i \in [0, 1), j \in [0, \frac{1}{2}), k \in [0, \frac{1}{2})$ with $k < \frac{1}{b+1}$ satisfying at least one of the following:

Z_1 $d_b(gx, gy, N_b) \leq i d_b(x, y, N_b)$

Z_2 $d_b(gx, gy, N_b) \leq j[d_b(x, gx, N_b) + d_b(y, gy, N_b)]$

Z_3 $d_b(gx, gy, N_b) \leq k[d_b(x, gy, N_b) + d_b(y, gx, N_b)]$

Then g has a unique fixed point.

Proof:
It follows from Theorem 2.10, Theorem 2.12 and Theorem 2.14.

Remark 2.17. Let (X, d_b) be a generalized N_b-metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = d_b(x, y)$, then Theorem 2.16 reduces to Zamfirescu's contraction in a generalized N-metric space(an analogue of Zamfirescu's contraction in metric space).

Theorem 2.18. Let X be a complete N_b-metric space and $g : X \to X$ a map for which there exist the real number, c satisfying $0 \leq c < \frac{1}{2}$ such that for each pair $x, y \in X$.

$$\int_0^{d_b(gx,gy,N_b)} \phi(t)dt \leq c \int_0^{d_b(x,y,N_b)} \phi(t)dt \tag{50}$$

where $\phi : [0, \infty) \to [0, \infty)$. Then g has a unique fixed point.
Proof:
Suppose g satisfies condition (32) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = g^n x_0$, then

$$\int_0^{d_b(x_{n+1},x_n,N_b)} \phi(t)dt = \int_0^{d_b(gx_n,gx_{n-1},N_b)} \phi(t)dt \tag{51}$$

$$\leq c \int_0^{d_b(x_n,x_{n-1},N_b)} \phi(t)dt \tag{52}$$

$$\leq c^n \int_0^{d_b(x_1,x_0,N_b)} \phi(t)dt \tag{53}$$

Setting $h_n = \int_0^{d_b(x_{n+1},x_n,N_b)} \phi(t)dt$, we have

$$h_n \leq ch_{n-1} \tag{54}$$

We then deduce that

$$h_n \leq ch_{n-1} \tag{55}$$

$$h_n \leq c^n h_0 \forall n \in N. \tag{56}$$

Using (N_3) of Definition 2.1, we obtain

$$d_b(x_i, x_N, N_b) \leq b\left[\sum_{i=1}^{N-1} d(x_i, x_{i+1}, N_b)\right] \tag{57}$$

$$= b\left[\sum_{i=1}^{N-1} h_i\right] \tag{58}$$

$$= b\left[\sum_{i=1}^{N-1} q^i h_0\right] \tag{59}$$

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Taking the limit of $d_b(x_i, x_N, N_b)$ as $N, i \to \infty$, we have

$$\lim_{n,m\to\infty} d_b(x_i, x_N, N_b) = \lim_{n,m\to\infty} b\left[\sum_{i=1}^{N-1} q^i h_0\right] = 0 \tag{60}$$

So, $\{x_n\}$ is a d_b-Cauchy Sequence.
By completeness of (X, d_b), there exist $x_o \in X$ such that x_n is d_b-convergent to x_o.
Suppose $gx_o \neq x_o$

$$\int_0^{d_b(x_n, gx_o, N_b)} \phi(t)dt \leq c\int_0^{d_b(x_{n-1}, x_o, N_b)} \phi(t)dt. \tag{61}$$

Taking the limit as $n \to \infty$ and using the fact that the function is d_b-continuous in its variables, we get

$$\int_0^{d_b(x_o, gx_o, N_b)} \phi(t)dt \leq c\int_0^{d_b(x_o, x_o, N_b)} \phi(t)dt = c\int_0^0 \phi(t)dt \tag{62}$$

A contradiction. So, $gx_o = x_o$.
To show the uniqueness, suppose $x_1 \neq x_2$ is such that $gx_1 = x_1$ and $gx_2 = x_2$ then

$$\int_0^{d_b(gx_1, gx_2, N_b)} \phi(t)dt \leq c\int_0^{d_b(x_1, x_2, N_b)} \phi(t)dt. \tag{63}$$

Since $gx_1 = x_1$ and $gx_2 = x_2$, we have

$$\int_0^{d_b(x_1, x_2, N_b)} \phi(t)dt \leq 0. \tag{64}$$

which implies that $x_1 = x_2$.

Remark 2.19. Let (X, d_b) be a generalized N_b-metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = d_b(x, y)$, then Theorem 2.18 reduces to general contractive condition of integral type in a generalized N-metric space(an analogue of general contractive condition of integral type in metric space).

3 Conclusions

In this paper, the notion of generalized N_b-metric spaces was introduced. This is a generalization of N-metric spaces, b-metric space, metric space, and many other spaces in literature. This newly introduced space considers the path integral in physics. The motivation comes from the expression of quantum mechanical amplitude for a particle to go from the initial point x to the final point y

$$\int Dqe^{(i/h)S(q)}.$$

Fixed point theorems satisfying some contractive conditions are stated and proved. This concept generalizes some known results in the literature.

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