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Historical Perspectives on Semigroup Theory: The Cold War Influence

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Abstract:
This paper covers another look on the history: growth, and development of semigroups during and after the cold war in the Union of Soviet Socialists Republic (USSR) and simultaneously in the Great Britain around the 1950's. The "Iron-curtain Mathematics" as I tag it.

1. Introduction:

The theory of semigroups is a relatively young branch of mathematics, with most of the major results having appeared after the Second World War. Semigroup theory might be termed 'Cold War mathematics' because of the time during which it developed. There were thriving schools on both sides of the Iron Curtain, although the two sides were not always able to communicate with each other, or even gain access to the other's publications. The term 'semigroup' was first coined in 1904 to provide a name for certain systems which were not groups but which arose during attempts to extend results on finite groups to the infinite case; the definition of these early 'semigroups', however, differed slightly from the modern notion. Hollings, C. (2009).

2.1. Definition: A binary operation or law of composition on a set G is a function, $G \times G \to G$ that assigns to each pair $(a, b) \in G \times G$ a unique element $a * b \text{ or } ab$ in G called the composition of $a \text{ and } b$. Howie, J. M. (1995)

2.2. Definition: A non-empty set equipped with one or more binary operations is called an algebraic structure. For example, $(\mathbb{Z}, +)$, $(\mathbb{N}, +)$, $(\mathbb{Q}, \div)$, etc. Howie, J. M. (1995)

2.3. Definition: A groupoid $(S,*)$ is defined as a non-empty set S on which a binary operation *— by which we mean a map $* : S \times S \to S$ — is defined. $(S,*)$ is a semigroup if the operation $*$ is associative, that is to say that for all $x, y \text{ and } z \text{ in } S$,

$$\big((x, y) *, z\big) * = (x, (y, z) *) *.$$

If $(x, y) *$ becomes $x.y$ or xy then equation (1) takes the form $(xy)z = x(yz)$, The normal associative law of elementary algebra. Further, when the semigroup is clear from the context, we use S rather than $(S,*)$. Howie, J. M. (1995)

3. Simultaneous Emergence of Semigroups During the Cold War.

This aspect of this paper on semigroups was mainly explored and expanded by the Russian schools of semigroups, which concerned generators, morphisms, ideals and congruences. Then 1966 Howie made remarkable results on idempotents transformation and in 1975 Sullivan initiated the study of $G(X)$-normal semigroups of transformation. However, The first 'proper' semigroup theory began to emerge in the 1920s with the work of the Russian mathematician Anton Kazimirovich Suschkewitsch who was doing (algebraic) semigroup theory before the rest of the world even knew there was such a thing! He was the first to prove a number of results that we now take for granted. For example, he proved that every semigroup may be embedded in a full transformation monoid (Suschkewitsch, 1926) — the semigroup analogue of Cayley's Theorem for groups. However, despite the publication of a textbook The Theory of Generalized Groups (1937), Suschkewitsch's work remained relatively unknown for many years, particularly in the West, and later researchers unwittingly rediscovered his results. The embedding of a (finite) semigroup in a full transformation monoid was reproduced by Stoll (1944), for example. The availability of Suschkewitsch's work cannot have been helped by the fact that most of the copies of his book were kept in Kharkov, a Ukrainian city which changed hands several times during the Second World War, with consequent destruction. Hollings, C. (2009).

The first 'proper' semigroup theory began to emerge in the 1920s with the work of the Russian mathematician Anton Kazimirovich Suschkewitsch as mentioned in the first part of this section. In fact, Suschkewitsch was doing (algebraic) semigroup theory before the rest of the world even knew there was such a thing! He was the first to prove a number of results that we now take for granted. For example, he proved that every semigroup may be embedded in a full transformation monoid (Suschkewitsch, 1926) — the semigroup analogue of Cayley's Theorem for groups. However, despite the publication of a textbook The Theory of Generalized Groups (1937), Suschkewitsch's work remained relatively unknown for many years, particularly in the West, and later researchers unwittingly rediscovered his results. The embedding of a (finite) semigroup in a full transformation monoid was reproduced by Stoll (1944), for example. The availability of Suschkewitsch's work cannot have been helped by the fact that most of the copies of his book were kept in Kharkov, a Ukrainian city which changed hands several times during the Second World War, with consequent destruction.

During the 1930s, the study of semigroups began to take off, although at this early stage was still heavily influenced by existing work on both groups and rings; semigroups were approached either by dropping selected group axioms, or by discarding an entire operation, namely addition, from a ring. As the decade progressed, the theory gradually gained momentum, culminating in the publication of three highly influential papers: Rees (1940), Clifford (1941) and Dubreil (1941) mentioned by Howie (2002). The paper of Dubreil, however, is rather different in character, and although it was no less influential than the other two, and was hailed by Clifford and Preston (1967, 174) as "ground-breaking", it would be slightly out of place in this work.

David Rees' 1940 paper contains semigroup theory's first major structure theorem, now known, appropriately enough, as the Rees Theorem. This result completed a strand of research initiated by Suschkewitsch (1928), and is analogous to the Wedderburn–Artin Theorem for rings. The structure theorem given by Clifford in his 1941 paper, however, has no analogue in either group or ring theory, and can therefore be taken to mark the beginning of an independent theory of semigroups.

Thanks to the influence of the above-mentioned papers, the theory of semigroups went from strength to strength in the 1940s, with an increasing number of papers appearing. Of course, the theory did not immediately emerge fully-formed, and so we find the following in the preface to Jacobson's Lectures in Abstract Algebra, in reference to semigroups: Though this notion appears to be useful in many connections, the theory of semigroups is comparatively new and it certainly cannot be regarded as having reached a definitive stage. (Jacobson,

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1951, 15) The 1950s saw the introduction of three broad concepts which continue to be of use and interest in the modern theory of semigroups: Green's relations, regular semigroups and inverse semigroups.

Moreover, the later aspects of the theory which will include Green's relations, first appeared in a paper by J. A. Green (1951); simply put, these are a collection of equivalence relations which may be defined within a given semigroup (in terms of its principal ideals) to enable us to study its 'large-scale' structure. Also introduced in Green's 1951 paper was the concept of a regular semigroup: a semigroup $S$is said to be regular if, for each $a \in S$, there exists $x \in S$such that $axa = a$. This notion was introduced for semigroups, at the suggestion of David Rees, by analogy with that of (Von Neumann) regularity in rings; the concept of a regular ring had been introduced by Von Neumann (1936) as an algebraic tool for the study of complemented modular lattices (Murray and von Neumann, 1936), which at that time were finding an application in the recasting of projective geometry in terms of lattices (Goodearl, 1981). The study of the various classes of regular semigroups has proved particularly fruitful over the years (Howie, 1995); perhaps the first of these to make an appearance was the class of completely regular semigroups in Clifford (1941).

The third and final concept that is important and must be mentioned here is that of an inverse semigroup. Inverse semigroups arose from the study of systems of partial one-one mappings of a set and the desire to find an abstract structure that corresponds to such a system, in much the same way that the study of permutations of a set had yielded the concept of an abstract group. As we have seen, they were introduced independently by Wagner in 1952, and by Preston in 1954. Wagner called these semigroups generalized groups; it was Preston who coined the term 'inverse semigroup'. An abstract inverse semigroup is defined to be a semigroup$S$in which every element $s \in S$has a unique generalized inverse, i.e., an element $so \in S$such that$ssos = s$ and $sosso = so$. Note that a traditional group inverse is an inverse in this generalised sense, but not conversely. This is just one of several equivalent ways of defining an inverse semigroup; we may, for example, define an inverse semigroup to be a regular semigroup whose idempotents (elements $e$with $e^2 = e$) commute (Howie,1995, Theorem 5.1.1). In the terminology to be used, an inverse semigroup is a regular semigroup in which the idempotents form a semilattice. The theory of inverse semigroups forms a major part of modern semigroup theory. Much more has been written on the history of inverse semigroups than on the history of any other aspect of semigroup theory. Howie, J. M. (1995).

It was also around this time that semigroup theory really began to take off in the USSR and, thanks to the relative lack of mathematical communication across the Iron Curtain, certain results in the growing theory of semigroups were duplicated in East and West. Inverse semigroups were a prime example of this, being introduced independently by Victor Vladimirovich Wagner (1952, 1953) in the Soviet Union, and by Gordon Preston (1954a, B.C) in Great Britain. As the 1950s progressed, each side was becoming more aware of the work of the other, and, at the very least, Russian papers were becoming more easily accessible in the West. Hollings, C. (2009).

4. Conclusion and Recommendation

The exploration of semigroup theory during and after the Cold War reveals a significant period of mathematical advancement influenced by geopolitical tensions. The Cold War era provided a unique environment that fostered rapid development in various scientific fields, including abstract mathematics. Semigroup theory, in particular, benefited from the increased focus on theoretical research driven by the need for technological superiority. This period saw the emergence of foundational concepts and theorems that continue to shape the field today.

Therefore, building on this historical foundation, it is recommended that contemporary researchers and educators emphasize the importance of historical context in mathematical development. Integrating historical case studies into the curriculum can provide students with a deeper understanding of how external factors influence mathematical progress. Additionally, fostering and encouraging international collaborations can help bridge gaps and promote further advancements in semigroup theory and other abstract mathematical fields. This approach will not only honor the legacy of Cold War mathematics but also inspire future generations to contribute to the ongoing evolution of the discipline.

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