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Numerical Scheme for the Solution of Fuzzy type Initial Value Problems by using of Fuzzy Laplace Transform

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Abstract
In this artical a nummerical scheme developed for approximate solution of the Fuzzy initial value problems (FIVPs) . The present numerical approximation scheme based on Fuzzy Laplace transform (FLT) and Trapezoidal rule (quadrature rule). In the construction of scheme we used hyperbolic contour to approximate Fuzzy inverse Laplace transform (FILT). Performance of the numerical scheme we developed checked by applying the scheme to Fuzzy IVPs. Results of our present scheme are compared with the results produced by different numerical method previously used by different researcher for solution of Fuzzy IVPs. The numerical experiments produce good results, which show supremacy of the present numerical scheme over other numerical methods.

Keywords: Fuzzy IVPs, Fuzzy Laplace transform, Fuzzy Inverse Laplace transform, Trapezoidal rule, Contour.

1 Introduction

Fuzzy differential equations (FDEs) play very important role in science and especially in engineering. Fuzzy differential equations utilized for mathematical modeling of physical phenomena like population model [5], medicine [1] and gravity [3]. Previously different researcher find the solution of FIVPs by different numerical as well as analytical methods . In [4] homotopy perturbation method used for the solution of FIVPs, [7] numerically solved FIVPs by Nystrom method and [6] used fifth order Runge-Kutta method for numerical approximation of FIVPs. Proposed numerical scheme is disscued in section 2 and in section 4 applicability of present method checked by solving FIVPs of 1st order.

Definition 1.1. (Fuzzy Initial Value Problem)
FIVPs is defined by

$$y\prime(t) = f(t, y(t)),\ t \,\epsilon\, [t_0, 1], \tag{1}$$
$$y(t_0) = y_0,\ \text{where } y_0 = [\underline{y_0}, \overline{y_0}],$$

and
$$y(t)_r = [\underline{y}(t, r), \overline{y}(t, r)],\ r \,\epsilon\, [0, 1].$$

Definition 1.2. (Laplace Transform) [9, p.449]
Laplace transform for given function $p(t)$ of a real variable t and $t \geq 0$ is represented as

$$\mathcal{L}[p(t)] = P(s) = \int_0^\infty e^{-st} p(t)dt, \quad Re(s) > 0, \tag{2}$$

*The author to whom all the correspondence should be addressed. Email:alizarshad988@yahoo.com here e^{-st} is the kernel of the transformation ,$s = x + i\sigma$ and s is transformed complex variable.

Definition 1.3. (Fuzzy Laplace Transform)
The FLT of a given Fuzzy function $y(t,r)$,$t \geq 0$ represented as

$$\mathcal{L}[y(t,r)] = Y(s,r) = [\ \int_0^\infty e^{-st}\underline{y}(t,r)dt, \int_0^\infty e^{-st}\overline{y}(t,r)dt]\ , \quad Re(s) > 0, \tag{3}$$

$$Y(s,r) = [\ \underline{Y}(s,r), \overline{Y}(s,r)]. \tag{4}$$

2 Numerical Scheme

Fuzzy IVP is given by

$$y\prime(t) = f(t, y(t)),\ t\ \epsilon\ [t_0, 1], \tag{5}$$
$$y(t_0) = y_0.$$

Applying Fuzzy Laplace transform to FIVP given in equation (5) we got

$$s\mathcal{L}\{y(t)\} - y(0)_r = F(s, Y(s)), \tag{6}$$

$$sY(s) - y(0)_r = F(s, Y(s)),\ Where\ \mathcal{L}\{y(t)\} = Y(s). \tag{7}$$

After simplification we got the following results

$$\underline{Y}(s,r) = \frac{\underline{y}(0,r)}{F(s)}, \tag{8}$$

and

$$\overline{Y}(s,r) = \frac{\overline{y}(0,r)}{F(s)}. \tag{9}$$

For inverse Laplace transform [9, p.450] of $\underline{Y}(s,r)$

$$\underline{y}(t,r) = \frac{1}{2\pi\ i}\int_{C-i\infty}^{C+i\infty} \underline{Y}(s,r)e^{st}\ \mathrm{d}s, \tag{10}$$

and for $\overline{Y}(s,r)$ we have

$$\overline{y}(t,r) = \frac{1}{2\pi\ i}\int_{C-i\infty}^{C+i\infty} \overline{Y}(s,r)e^{st}\ \mathrm{d}s. \tag{11}$$

2.1 Approximations of Path(Path of integration)

The ILT of some function is not easy to calculated and in some case even not exist. In this paper we use concept of contour integration and choose a contour for the approximation of the line $C - i\infty\ \ to\ \ C + i\infty$. For the approximation of integration path we have to select a contour which give us precise result. The one Contour is parabolic and the other one is hyperbolic [8, 10].
Parametric presentation of hyperbolic is given by

$$s = \omega + \lambda\ (1 - sin(\sigma -\ i\ u)), \qquad -\infty <\ u\ < \infty, \tag{12}$$

in this paper we will used hyperbolic contour.

$$\underline{y}(t,r) = \frac{1}{2\pi\ i}\int_\Gamma \underline{Y}(s,r)e^{st}\ \mathrm{d}s, \tag{13}$$

and

$$\overline{y}(t,r) = \frac{1}{2\pi\ i}\int_\Gamma \overline{Y}(s,r)e^{st}\ \mathrm{d}s. \tag{14}$$

Where the Γ represent the left branch of hyperbolic contour of integration given in (12) then equation (13) and (14) become

$$\underline{y}(t,r) = \frac{1}{2\pi\ i}\int_{\Gamma} \underline{Y}(s(u),r)e^{s(u)t}s'(u)\ \mathrm{d}u, \tag{15}$$

$$\overline{y}(t,r) = \frac{1}{2\pi\ i}\int_{\Gamma} \overline{Y}(s(u),r)e^{s(u)t}s'(u)\ \mathrm{d}u, \tag{16}$$

where $Y(s(u)) = Y(s)$ and use equal weight quadrature rule (tarpezodial rule) with $k > 0$ here we set take $s_j = z(u_j), s'_j = s'(u_j)$ the equations (15) and (16) can be approximated as

$$\underline{y}_N(t,r) = \frac{k}{2\pi\ i}\sum_{j=-N}^{N} \underline{y}(s_j)e^{s_j t}s'_j, \tag{17}$$

and

$$\overline{y}_N(t,r) = \frac{k}{2\pi\ i}\sum_{j=-N}^{N} \overline{y}(s_j)e^{s_j t}s'_j. \tag{18}$$

Theorem 2.1. [8] Let y is solution of (5) and $\hat{f}$ is analytic in Σ^{ω}_{β}.Let $0 < t_0 < T$,$0 < \theta < 1$, and let $b > 0$ is given as $\cosh b = \frac{1}{(\theta\tau\sin\sigma)}$ where $\tau = t_0/T$.Let r satisfy $0 < r < min(\sigma, \beta - \pi/2 - \sigma)$ so that $\Gamma \subset S_r \subset \Sigma^{\omega}_{\beta}$, and let the scaling factor be $\lambda = \theta\tilde{r}N/(bT)$.Therefore, we have for the approximate solution $y_N(t)$ defined by (17) and (18) with $k = b/N \le \tilde{r}/\log 2$. $||y_N(t) - y(t)|| \le CMe^{\omega t}\ l(\rho_r N)\ e^{-\mu N}\ (||y_0|| + ||\hat{f}||_{\Sigma^{\omega}_{\beta}})$, for $t_0 \le t \le T$, where $\mu = \tilde{r}(1-\theta)/b$,$\rho_r = \theta\tilde{r}\tau\sin(\sigma - r)/b$, and $C = C_{\sigma,r,\beta}$.

3 Data Use

Following data is used in the numerical experiments T=2,$t_0 = 0.01$ $\theta = 0.1$, $\sigma = 0.3812$, $\tau = (t0/T)$, $b = \cosh^{-1}(1/(\theta\tau\sin(\sigma)))$, $\varphi = 0.3431, \tilde{\varphi} = 2\pi r$, $k = b/N$, $w = 02$, $\lambda = (\theta\tilde{\varphi}N)/(bT)$, $E_1 = |\underline{y}(t,r) - \underline{y}_N(t,r)|$, $E_2 = |\overline{y}(t,r) - \overline{y}_N(t,r)|$ and $Eb = Error\ bound$

4 Numerical Experiments

Experiment 1 [2] Let us consider the Fuzzy IVPs

$$y\prime(t) = y(t),\ t\ \epsilon\ [0,1], \tag{19}$$

$$y(0) = (0.8 + 0.125r, 1.1 - 0.1r),\ r\ \epsilon\ [0,1].$$

The exact solution at t=1 is

$y(1,r) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e)$, where $\underline{y}(1,r) = (0.8 + 0.125r)e\ and\ \overline{y}(1,r) = (1.1 - 0.1r)e$.

Solution:

After applying Fuzzy Laplace transform (19), we got the result

$\underline{Y}(s,r) = \frac{0.8+0.125r}{s-1}\ and\ \overline{Y}(1,r) = \frac{1.1-0.1r}{s-1}$, where $\underline{Y}(s,r), \overline{Y}(s,r)$ are the Laplace transform of $\underline{y}(t,r), \overline{y}(t,r)$ respectively. Applying the present numerical scheme, we got result given in table1 at $t = 1$ and $N = 200$ (Number of steps).

Experiment 2 [4]

$$y\prime(t) = 2.y(t) + (t^2 + 1),\ t\ \epsilon\ [0,1], \tag{20}$$

$$y(0) = (r, 2 - r),\ r\ \epsilon\ [0,1].$$

The exact solution of (21) is

$y(t,r) = ((r+\frac{3}{4})e^{2t} - \frac{1}{4}(2t^2+2t+3), (\frac{11}{4}-r)e^{2t} - \frac{1}{4}(2t^2+2t+3))$, where $\underline{y}(t,r) = (r+\frac{3}{4})e^{2t} - \frac{1}{4}(2t^2+2t+3)$, *and* $\overline{y}(t,r) = (\frac{11}{4} - r)e^{2t} - \frac{1}{4}(2t^2 + 2t + 3)$.

Solution:

Applying Fuzzy Laplace transform (21), we have

$\underline{Y}(s,r) = \frac{(r+s^3+s^2+2)}{s^3(s-2)}$ *and* $\overline{Y}(s,r) = \frac{((2-r)s^3+s^2+2)}{s^3(s-2)}$. Using the present numerical scheme, we have the result given in table 2 at $t = 0.1, 0.3$.

Experiment 3 [4]

$$y\prime(t) = -3y(t) + e^t,\ t\ \epsilon\ [0,1], \tag{21}$$

$$y(0) = (r-1, 1-r),\ r\ \epsilon\ [0,1].$$

The exact solution of (**??**) is

$y(t,r) = ((r-\frac{5}{4})e^{-3t} + \frac{1}{4}e^t, (\frac{3}{4}-r)e^{-3t} + \frac{1}{4}e^t)$, where $\underline{y}(t,r) = (r-\frac{5}{4})e^{-3t} + \frac{1}{4}e^t$ *and* $\overline{y}(t,r) = (\frac{3}{4}-r)e^{-3t} + \frac{1}{4}e^t$.

Solution:

Applying Fuzzy Laplace transform (**??**), we have

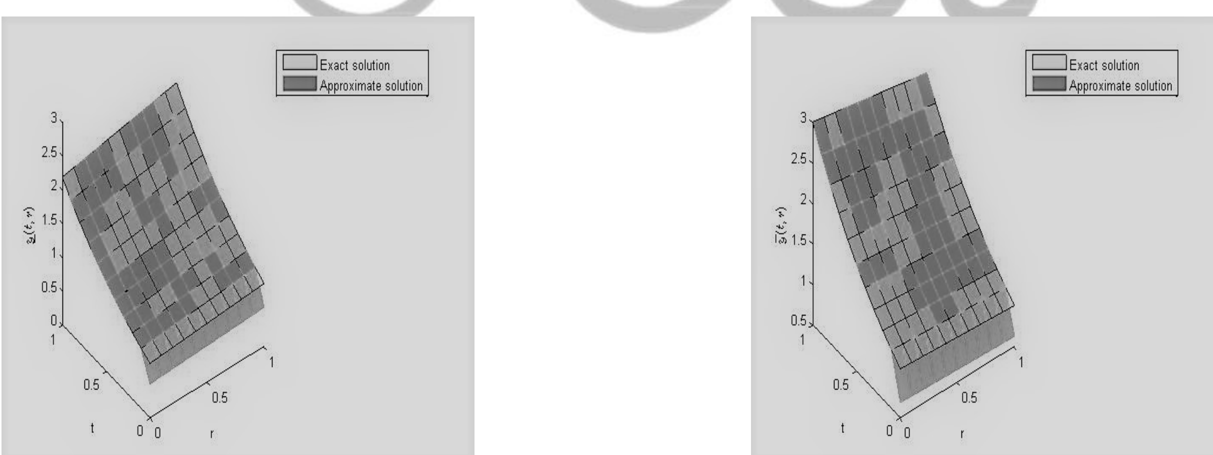
$\underline{Y}(s,r) = \frac{(r-1)}{(s+3)} + \frac{1}{(s+3)(s-1)}$ *and* $\overline{Y}(s,r) = \frac{(1-r)}{(s+3)} + \frac{1}{(s+3)(s-1)}$. Using the present numerical scheme, we have the result given in table 3 at $t = 0.1, 0.3$ and $r = 0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5\ 0.6\ 0.7\ 0.8\ 0.9\ 1$.

r	E_1	E_2	Eb
0	1.4388e-13	1.9718e-13	4.9290e-18
0.2	1.4699e-13	1.9407e-13	
0.4	1.5277e-13	1.9007e-13	
0.6	1.5765e-13	1.8607e-13	
0.8	1.6209e-13	1.8208e-13	
1	1.6609e-13	1.7852e-13	

Table 1: Absolute error between numerical solution of (19) using present numerical schemes and exact solution for N=200.

	t=0.1		t=0.3		t=0.1	t=0.3
r	E_1	E_2	E_1	E_2	Eb	
0	1.2213e-15	1.1102e-14	3.2197e-15	3.0198e-14	4.9290e-18	4.9290e-18
0.1	1.8041e-15	1.1102e-14	4.6629e-15	2.8422e-14		
0.2	2.2205e-15	9.7699e-15	5.8842e-15	2.8413e-14		
0.3	2.5535e-15	9.3259e-15	6.9932e-15	2.6646e-14		
0.4	3.3306e-15	8.8818e-15	8.8818e-15	2.5757e-14		
0.5	3.5507e-15	8.4378e-15	9.9920e-15	2.3537e-14		
0.6	3.8858e-15	7.5495e-15	1.1546e-14	2.2205e-14		
0.7	4.4409e-15	7.9936e-15	1.2657e-14	2.0872e-14		
0.8	4.8850e-15	7.1055e-15	1.4211e-14	1.9096e-14		
0.9	5.9952e-15	6.6613e-15	1.5543e-14	1.7319e-14		
1	6.2172e-15	6.2125e-15	1.5987e-14	1.5987e-14		

Table 2: Absolute error between numerical solution of (21) using present numerical schemes and exact solution for N=200.

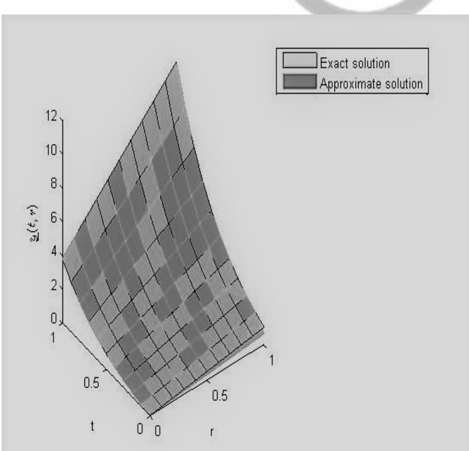


(a) (b)

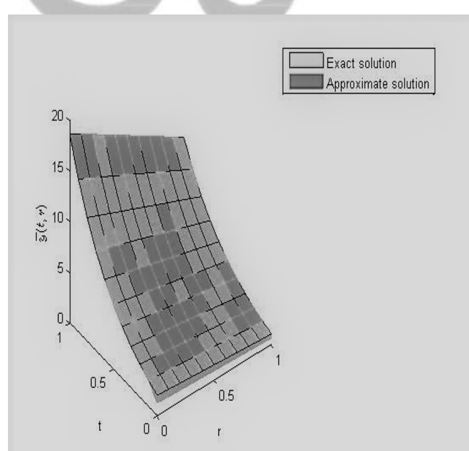
Figure 1: (a) Comparision between the exact soltion and the approximate solution of $\underline{y}(t,r)$ given in Experiment 1 . (b) Comparing the exact and the approximate solution of $\overline{y}(t,r)$ given in Experiment 1.

	t=0.1		t=0.3		t=0.1	t=0.3
r	E_1	E_2	E_1	E_2	Eb	
0	9.9924e-16	1.8874e-15	3.5810e-15	6.1063e-15	4.9290e-18	4.9290e-18
0.1	8.8846e-16	1.6653e-15	3.1365e-15	5.4402e-15		
0.2	6.6639e-16	1.7767e-15	2.5951e-15	4.9960e-15		
0.3	5.5584e-16	1.3323e-15	2.1441e-15	4.5519e-15		
0.4	3.8908e-16	1.5544e-15	1.6411e-15	4.2189e-15		
0.5	1.6729e-16	1.1657e-15	1.1868e-15	3.6082e-15		
0.6	1.1106e-16	1.0547e-15	6.9390e-16	3.2198e-15		
0.7	5.5514e-17	9.4370e-16	2.0880e-16	2.6646e-15		
0.8	2.0828e-16	7.2165e-16	2.4983e-16	2.2204e-15		
0.9	3.2963e-16	6.3838e-16	7.7716e-16	1.6653e-15		
1	4.7190e-16	4.7190e-16	1.2212e-15	1.2212e-15		

Table 3: Absolute error between numerical solution of (**??**) using present numerical schemes and exact solution for N=200.

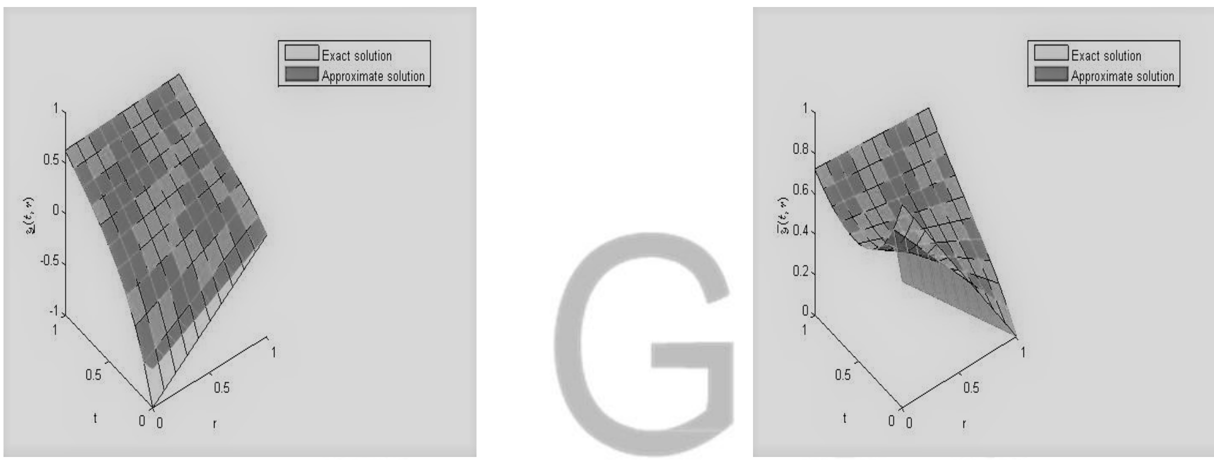


(a)



(b)

Figure 2: (a) Comparision between the exact and the approximate solution of $\underline{y}(t,r)$ given in Experiment 2. (b) Comparing the exact and the approximate solution of $\overline{y}(t,r)$ given in Experiment 2.



(a) (b)

Figure 3: (a) Comparision between the exact and the approximate solution of $\underline{y}(t, r)$ given in Experiment 3. (b) Comparing the exact and the approximate solution of $\overline{y}(t, r)$ given in Experiment 3.

5 Conclusion

The numerical experiment show the efficiency of the present numerical algorithm.The proposed method is useful and easy to apply not only to FIVPs but also to Fuzzy Volterra integral equation. Especially for the Fuzzy Volterra integral equation involving convolution of functions.

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