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RECENT ADVANCES IN APPLICATIONS OF GRAPH THEORY

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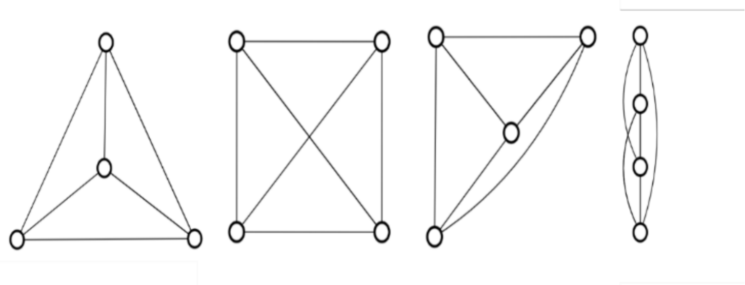
***Abstract*—** The field of mathematics plays vital role in various fields. One of the important areas in mathematics is Graph theory. The era of graph theory began with Euler in the year 1735 to solve the well-known problem of the Königsberg Bridge. In the modern age, graph theory is an integral component of computer science, artificial engineering, machine learning, deep learning, data science, and social networks. Modern Applications of Graph Theory discusses many cutting-edge applications of graph theory, such as traffic networks, navigable networks and optimal routing for emergency response, and graph-theoretic approaches to molecular epidemiology. In this paper describe the description and recent applications of graph theory.

***Keywords*—** Algorithm, Connectivity, Constraints, Coloring, Graph theory, social network analysis.

1. Introduction

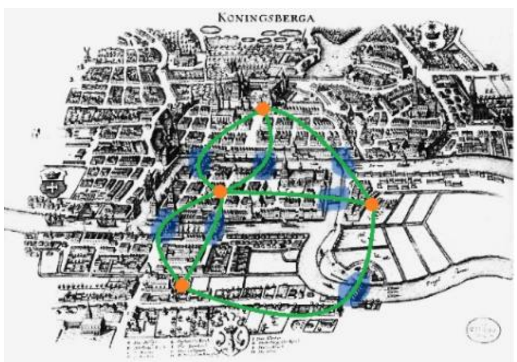
Graph theory is a branch of discrete mathematics with the study of relationships between objects. These objects are represented as dots and the relationships as lines. The dots are called vertices or nodes, and the lines are called edges or links. The connection of all the vertices and edges together is called a graph and can be represented as an image, like the ones below:

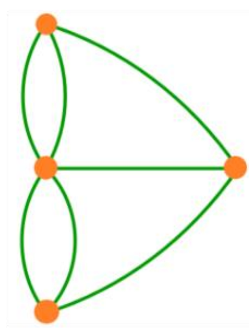


One of the most important properties of graphs is that they are just abstractions of the real world. This allows the representation of graphs in many different ways, all of which are correct. For example, all the graphs above are visually very different from each other; however, they all represent the same relations, thus all are the same graph. You can check it by counting the number of edges that each vertex has.

2.The origin of graph theory

Königsberg (now Kaliningrad, Russia) was a city from the old Kingdom of Prussia spanning along both sides of the Pregel river. The city had two islands that were connected to the mainland through bridges. The smaller island was connected with two bridges to either side of the river, while the bigger island was connected with only one. Additionally, there was one bridge connecting both islands. You can see the layout of the bridges in the image below.





Imagine that if you are a tourist and want to cross all 7 bridges because they are the main attraction of the city. However, you are a bit lazy and do not want to walk too much. So, you do not want to cross the same bridge more than one time. Is there a path through the city that does this? Just as a simple rule, you can only cross the river through bridges, so no swimming. How would you solve this problem?

Euler recognized that the problem was not about measuring and calculating the solution, but about finding the geometry and relations behind it. By abstracting the problem, he started the field of graph theory and his solution became the first theorem of this field. Since then, graph theory has developed not only from a mathematical perspective, but into many other fields such as physics, biology, linguistics, social sciences, computer sciences and more.

To analyse the graph theory applications two problems areas are considered,

1- Classical problems and
2- Problems from the real-life applications

The Classical problems are defined with the help of the graph theory as connectivity. A graph G(V, E) is a non-linear data structure, which consists of pair of sets (V, E) where V is the non-empty set of vertices (points or nodes). E is the set of edges (lines or branches) such that there is a mapping f: E →V i.e., from the set E to the set of ordered or unordered pairs of elements of V. The number of vertices is called the order of the graphs and the number of edges is called the size of graph G (V, E).

Graphs are of three types based on their properties, Undirected Graphs, Directed graphs, and weighted graphs.

2.1 Undirected Graph

In Undirected Graphs, the edges are associated with an unordered pair of vertices. A graph G (V, E) without a loop and parallel edges is called a simple graph. A graph that has more than one edge between any pair of vertices is called a multigraph. Again, if any multigraph contains loops, then the graph is a Pseudo graph. According to structure, there are different types of undirected graphs, such as Null graphs, complete graphs, Regular graphs, bipartite graphs, Cycles, Wheels, Eulerian graphs, and Hamiltonian graphs.

2.2 Directed Graph

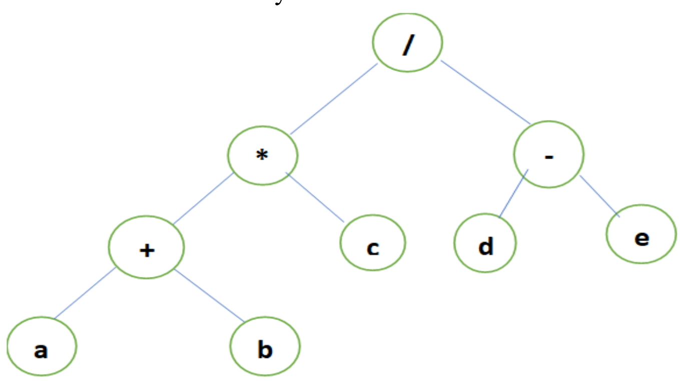
A directed or digraph graph G consists of a set V of vertices and a set E of edges such that e ϵ E is associated with an ordered pair of vertices i.e., each edge has a direction. There are different types of directed graphs. Symmetric directed graphs, simple directed graphs, complete directed graphs, quasi-transitive digraphs, and oriented graphs. For example, in a graph of a social network, where vertices represent people and edges connect people that have a relationship. These relationships go both ways.

2.3 Weighted Graph

Many graphs can have edges containing a weight associated to represent real-world implications such as cost, distance, and quantity. Weighted graphs could be directed or undirected graphs.

2.4 Tree

Trees are one of the most commonly used sub-categories of graphs. In computing, trees are useful for organizing and storing data in a database. A tree is a connected acyclic graphic with no cycle. A tree T with n vertices has n-1 edges. A subgraph T a connected graph G (V, E) is called a spanning tree, if T is a tree and it includes every vertex of G. There are two most useful algorithms a). BFS (Breadth-first search) and b). DFS (Depth-first Search) for constructing the spanning trees of a given undirected graph G. For weighted graphs one can construct the minimal spanning tree using Prim's algorithm and Kruskal's algorithm. The Binary trees having one vertex of degree two and the other vertices of degree one or degree three, are used to represent an algebraic expression and storage representation. Storage Representation of Binary tree has two ways a) Sequential representation and b) Link representation.



Example: Use a binary tree to represent the expression **((a + b)* c) / (d - e)**

3 The real-life applications of graph theory

In this section, we will discuss some of the application problems of graph theory with real life examples. The calculation of their solution can be done with a variety of algorithms that will encourage the reader to look up. Moreover, the solutions of such problems may not be unique nor exact. Graph theory algorithms depend on the size and complexity of the graph; this means that some solutions may just be a very good approximation to the exact solution. Even more, some problems have not even been solved, thus approximations are the best outcome.

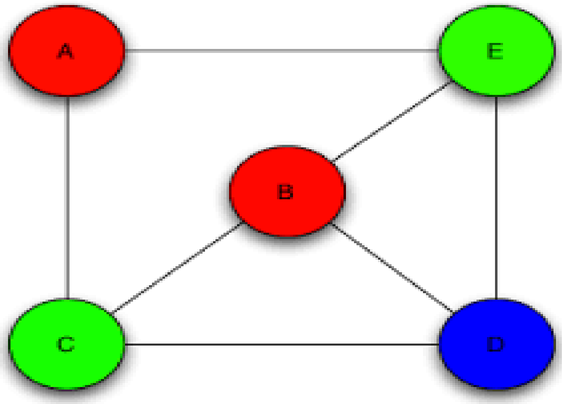
The study of graphs across a structure provides answers to numerous problems, it has developed not only from a mathematical perspective, but into many other fields such as physics, biology, linguistics, social sciences, computer sciences, layout, networking, optimization, matching, and operation research and many more.

4 Graph Coloring Problem

Graph colouring is one of the most useful techniques in which adjacent vertices obtain different colours. The minimum number of colours used for the correct colouring of the graph is our goal which is an optimization problem.
The problem of graph colouring has many applications, such as Making a Schedule or Time Table, Mobile Radio Frequency Assignment, Sudoku, Register Allocation, and Map Colouring.

5. Chromatic Number

The smallest number of colours needed to colour a graph G is called its chromatic number. For example, the following graph coloured minimum 3 colours.



6. Time Table or Scheduling Problem

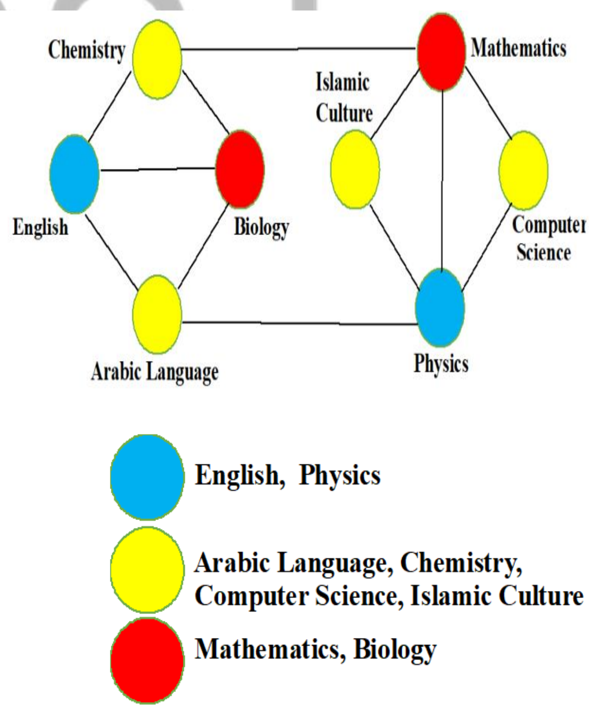
To think about a specific semester in a school; there are students taking each of the following combinations of topics. In this problem, our aim is to find the minimum number of examination days for scheduling the examination in the 8 subjects so that students taking any of the given combinations of the subject have no conflict.

In addition, find an available schedule using a minimum number of days.

Table: Combinations of Subjects

Courses	Subject 1	Subject 2	Subject 3	Subject 4
1	Math	Chemistry	Biology	English
2	Physics	Arabic Language	Biology	English
3	Biology	English	Arabic Language	-
4	Computer Science	Physics	Math	-
5	Islamic Culture	Math	Chemistry	-
6	Islamic Culture	Physics	Math	-
7	Computer Science	Math	Physics	-
8	Chemistry	Biology	English	-
9	Computer Science	Physics	-	-
10	Islamic Culture	Physics	-	-

The outcome of the problem



7. Airline Scheduling (Flow problem)

One of the most popular applications of graph theory falls within the category of flow problems, which encompass real life scenarios like the scheduling of airlines. Airlines have flights all around the world and each flight requires an operating crew. Personnel might be based on a particular city, so not every flight has access to all personnel. In order to schedule the flight crews, graph theory is used.

For this problem, flights are taken as the input to create a directed graph. All serviced cities are the vertices and there will be a directed edge that connects the departure to the arrival city of the flight. The resulting graph can be seen as a network flow. The edges have weights, or flow capacities, equivalent to the number of crew members the flight requires. To complete the flow network a source and a sink vertex have to be added. The source is connected to the base city of the airline that provides the personnel and the sink vertex is connected to all destination cities.

Using graph theory, the airline can then calculate the minimum flow that covers all vertices, thus the minimum number of crew members that need to operate all flights. Additionally, by giving weights to the cities corresponding to its importance, the airline can calculate a schedule for a reduced number of crew members that do not necessarily visit all the cities.

This flow problem can also be applied to many other instances. For example, when having to supply stores from warehouses with a finite number of trucks, or when scheduling public transport in specific routes considering the expected amount of people that will be using it.

8. Directions in a map (Shortest path)

Nowadays, we use our smart phones all the time to help us in our everyday lives. For me, it helps me by giving me directions to cycle from my location to a restaurant or a bar. But how are these directions calculated? Graph theory is the answer for this challenge, which falls in the category of defining the shortest path.

The first step is to transform a map into a graph. For these all-street intersections are considered as vertices and the streets that connect intersections as edges. The edges can have weights that represent either the physical distance between vertices, or the time that takes to travel between them. This graph can be directed showing also the one-way streets in the city.

Now, to give the direction between two points in the map, an algorithm only needs to calculate the path with the lowest sum of edge weights between the two corresponding vertices. This can be trivial for small graphs; however, for graphs created from big cities, this is a hard problem. Fortunately, there are many different algorithms that may not give the perfect solution, but will give a very good approximation, such as the Dijkstra's algorithm or the A* search algorithm.

Finding the shortest or fastest route between two points in the map is definitely one of the most used applications of graph theory. However, there are other applications of the shortest path problem. For example, in social networks, it can be used to study the “six degrees of separation” between people, or in telecommunication networks to obtain the minimum delay time in the network.

9. Solving Sudoku’s puzzle (Graph coloring)

Sudoku is a popular puzzle with a 9x9 grid that needs to be filled with numbers from 1 to 9. A few numbers are given as a clue and the remaining numbers needed to be filled follow a simple rule: they cannot be repeated in the same row, column or region. This puzzle, despite using numbers, is not a mathematical puzzle, but a combinational puzzle that can be solved with the help of graph colouring.

One can convert the puzzle to a graph. Here, each position on the grid is represented by a vertex. The vertices are connected if they share the same row, column or region. This graph is an undirected graph, since the relationship between vertices goes both ways. An important feature of the graph is the assignment of a label to each vertex. The label corresponds to the number used in that position. In graph theory, the labels of vertices are called colours.

To solve the puzzle, one needs to assign a colour to all vertices. The main rule of Sudoku is that each row, column or region cannot have two of the same numbers, thus two vertices that are connected cannot have the same colour. This problem is called graph colouring, and, as with other graph theory problems, there are many different algorithms that can be used to solve this problem for example Greedy colouring or Greedy knapsack algorithm, but their performance depends highly on the graph itself.

The colouring problem is used normally for very fundamental problems. However, there are more real-life problems that can be translated to a colouring problem, such as scheduling tasks. For example, scheduling exams in rooms. Each exam is a vertex and there is an edge connecting them if it takes place at the same time. The graph created is called an interval graph, and by solving the minimum colouring problem of the graph, you obtain the minimum number of rooms needed for all the exams. This can be generalized with tasks that use the same resources, such as compilers of programming languages or bandwidth allocation to radio stations.

10. Search Engine or PageRank algorithm

Search engines such as Google let us navigate through the World Wide Web without a problem. Once a query is made to search a specific set of words, the engine looks for websites that match the query. After finding millions of matches, how does the engine rank them to show the most popular ones first?

The search engine solves this through graph theory by first creating a web graph, a graph where the vertices are the websites, and the directed edges follow hyperlinks within those websites. The result is a directed graph that shows all relations between websites. Additionally, one can add weights to the vertices to give priority to more important or influential websites.

To classify the most popular websites, different algorithms can be used. One of the first ones used by Google is called Page Rank. Here, the engine assigns probabilities to click a hyperlink and iteratively adds them up to form a probability distribution. This distribution represents the likelihood of a person randomly arriving at a particular website. Then, the engine orders the list of websites according to this distribution and shows the highest ones.

This algorithm had many faults. One can exploit it by having for example blog websites with many links to a particular website to increase the click probability, or by buying hyperlinks in websites with higher weights. Nowadays, there are more complicated algorithms that also consider sponsored advertisement, but the main core is still graph theory and the relations between websites.

11. Social Media Marketing (Community detection)

In January 2022, Facebook had 2.9 billion active users. As a social media platform, most of the revenue comes from adverting. Having so many users, advertisers will find it very expensive to place their advertising campaigns within the reach of everyone. However, one can also just target the people that may be interested in your product. How can you define such a target audience?

Using graph theory, you can create a social network graph by assigning a vertex to each person. You connect vertices with edges if the persons have a relationship, such as friends in Facebook. This leads to an undirected graph. This massive graph would appear at the beginning very chaotic; however, one can always find patterns in it.

A way to find the ideal target audience is to decompose the graph into smaller sub-graphs. There are different algorithms that can do this, such as hierarchical clustering algorithms or minimum cut methods like the Karger's algorithm. The result is the division of the graph into clusters of people that are highly connected to each other, but less connected to other groups of people. These groups are called communities and they share common interests, like specific artists, brands or even political parties. Identifying these communities is advantageous for advertising since they are more likely to buy common products, follow similar artists or vote for similar parties.

The detection of communities can be also used for other purposes than advertisement. After identifying the communities, one can compare connections between groups or even within groups. If a group or a vertex within the group does not behave as their peers, it can be a sign of intrusion. This can be used as a security control. For example, if the vertices are computers or programs in a network, strange behaviour could be caused by attacks on it. Identifying strange connections can improve the security of the network.

Conclusion

The main aim of this paper is to present the importance of graph theoretical ideas in various areas of computer applications. This paper is designed to benefit the students of computer science to gain depth knowledge on graph theory and its relevance with other subjects like operating systems, Networks, Databases, software engineering etc. this paper focused on the various applications of major graph theory that have relevance to the field of computer science and applications this will explore the usage of graph theory in data science and related fields in which it has become a very popular area of research in recent years.

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