Global Scientific JOURNALS

GSJ: Volume 13, Issue 1, January 2025, Online: ISSN 2320-9186
www.globalscientificjournal.com

UNVEILING THE MYSTERY OF THE COLLATZ CONJECTURE

Sandoval Amui

(Independent researcher. M.Sc. degree in petroleum engineering from Louisiana State University. Graduate courses in offshore engineering from The University of Texas at Austin).

Abstract

The Collatz Conjecture is a puzzle often listed among the top ten unsolved math problems, and reputed to be an unsolvable or undecidable conundrum, which yields sequences of numbers of chaotic and unpredictable nature. Some experts say it is a siren song, or a quagmire, while others believe it is a problem out of reach of current mathematics. Its apparent simplicity has a great seductive power, but experts often advise amateurs to stay away from the challenge that has frustrated the best world's mathematicians, since its proposition in 1937. Contrarily to the prevailing views, I say the Collatz Conjecture is not an unsolvable or undecidable puzzle, and that there is no mystery in it, only elementary arithmetic. The two operating rules imposed by the problem imply a unique arrangement of numbers, which is the arithmetic representation of Collatz Conjecture, and reveals its essence. Under the rationale behind that unique arrangement, the meaning of infinity becomes irrelevant. The sequences built under the Conjecture rules follow a chain of sections of different geometric progressions of integers, which results from an ingenuous combination of the effects of these two arithmetic rules. This arithmetic mechanism accounts for the supposed secret hidden in the Conjecture.

Key words: Conjecture; Collatz; 3n + 1 Problem; Syracuse algorithm; Arrangement; Geometric progressions; Convergence; Sequence; Infinity.

Introduction

The Collatz Conjecture (the "Conjecture") is an old and famous math problem, which has defied mathematicians and non-mathematicians for many decades. Its origin is uncertain, and often attributed to the German mathematician Lothar Collatz, who proposed that question in 1937. The Conjecture is also known by different names, as the "3n + 1" Problem, the Syracuse Algorithm, and others. The reader may find plenty of information on that famous problem on the internet.

A curious aspect of said old math question is the fact that, although being simply stated and easily understood, it remains unproven, what makes it an appealing challenge to beginners. Take any integer of your free choice: in case of an even number, divide it by two; in case of an odd number, triple it and add one. The Conjecture says that if you keep applying these two simple rules to any subsequent number you will create a sequence of numbers (the "Sequence"), which eventually reach the loop "1, 4, 2, 1", whatever the initial number.

There are many discouraging references to it as being unsolvable, impossible or undecidable, as well as many advices to comers to stay away from that dangerous question, which has resisted to almost any famous mathematician. We often see that Conjecture listed among the top ten hardest mathematical problems, and statements that mathematics may not be ready for it.

As a relevant fact, all tests performed to verify said math problem, including the use of algorithms and modern computers, indicated that the Conjecture is a true conjecture. Any interested person may find some Python-based programs on the internet, which yield Sequences for any initial number. To date, no counter example occurred. There remains three possibilities: the Conjecture is true, false or undecidable.

In this book, I offer a comprehensive analysis of the Conjecture with the purpose to explain its arithmetic meaning, and show how the Sequences behave and why they end in number "1". I am convinced it is true and there is no mystery behind this math puzzle, only elementary arithmetic. Contrarily to the present understanding, that the Conjecture is unsolvable, undecidable or whatever, and that its Sequences are number arrangements of chaotic and unpredictable nature, I say the two Conjecture rules imply a unique arrangement of numbers (the "Arrangement") and impose a convergence mechanism, which takes all Sequences to number "1". The rationale behind that unique Arrangement overcomes the difficulty to deal with the meaning of infinity.

In building that unique Arrangement, I combined different geometric progressions of a common ratio "2" with an arithmetic progression of natural odd numbers from "1" to infinity, and explained the real nature of the Conjecture. The Arrangement contains all integers from "1" to infinity, without repetition of any number. I am convinced that my approach with the help of said unique Arrangement and the use of geometric progressions contradict the generalized feeling that the Sequences built under the Conjecture are number arrangements of chaotic and unpredictable nature, and that a formal proof of the Conjecture is impossible.

The convergence of the Sequences occurs because they move in accordance with sections of different geometric progressions, under the rule "n/2". The rule "3n + 1" avoids the fractional number that the rule "n/2" would generate after each section reaches an odd number, by making the Sequence to leap into another section of a different geometric progression of a common ratio "1/4". Within each section, the rule “n/2” modifies that common ratio “1/4” to “1/2”. That process goes on until the Sequence reaches a power of number “4” in the specific geometric progression that ends in number “1” and reaches number “1” through said rule “n/2”.

The present analysis explains the arithmetic meaning and the structure of the Conjecture, as well as how the Sequences behave and why they end in number “1”. I will show that the Conjecture in fact uses a clever arithmetic mechanism to accomplish the task to make any Sequence to reach number “1” by following sections of different geometric progressions of a common ratio “1/4” (generated by the rule “3n + 1”), converted to a common ratio “1/2” (under the effect of the rule “n/2”).

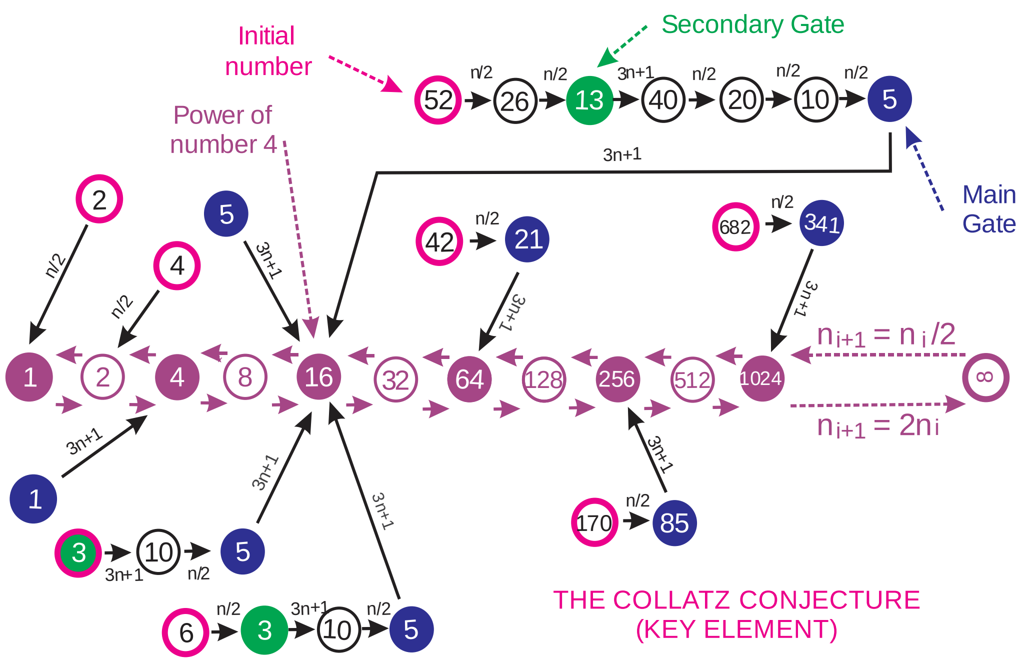
The Conjecture yields an infinite number of theoretical infinite Sequences, each one ruled by an odd number, which start in infinity (whatever infinity means) and converge to number “1”. Under the Conjecture rules, the Sequences do not go to infinity. They come from infinity. In practical terms, we start with finite numbers, even very large numbers, creating an infinite quantity of subsequences of each theoretical infinite Sequence, once we never reach infinity.

Before discussing the referred Arrangement and the convergence mechanism, it is convenient to introduce some observable facts about the Conjecture.

Observable facts

There are some relevant facts about the Conjecture we observe in all practical examples so far performed, as illustrated in Figure 1:

Figure 1: Key element



Whatever the initial number (circles in pink, as “6”, “42”, “170”, …), the resulting Sequence will converge to number “1” by previously reaching a term in a specific geometric progression of a common ratio "4" built as from number "1", here referred as the "Main Even Sequence" (powers of number "4", as 4 = 4^1, 16 = 4^2, 64 = 4^3, 256 = 4^4, …). In the Main Even Sequence, only the terms of said geometric progression of a common ratio "4" can yield odd integers under the inverse of the rule "3n + 1", here named "Main Gates".

All Sequences (other than the Sequences that start with a number in the Main Even Sequence) reach a power of number "4" in the Main Even Sequence through a Main Gate, what means an odd number, to which we apply the rule "3n + 1" to obtain a power of number "4". After joining a power of number "4" in the Main Even Sequence, all Sequences remain in that Main Even Sequence and converge to number "1" under the rule "n/2".

Some Sequences go directly to a Main Gate (circles in blue, as "5", "21", "85" …) and then directly to the Main Even Sequence (in purple). All other Sequences go to a different odd number, here referred as "Secondary Gate" (circles in green, as "3", "13", …) before reaching a Main Gate, what means an indirectly connection to the Main Even Sequence.

There is a clear indication that the Main Even Sequence is a key element to understand and explain the Conjecture, and I will show why it happens with the help of the Arrangement, which is consistent with the Conjecture rules, by taking the Main Even Sequence as a model to build the other Even Sequences from "3" to infinity. The Arrangement is the arithmetic representation of the Conjecture.

The Arrangement

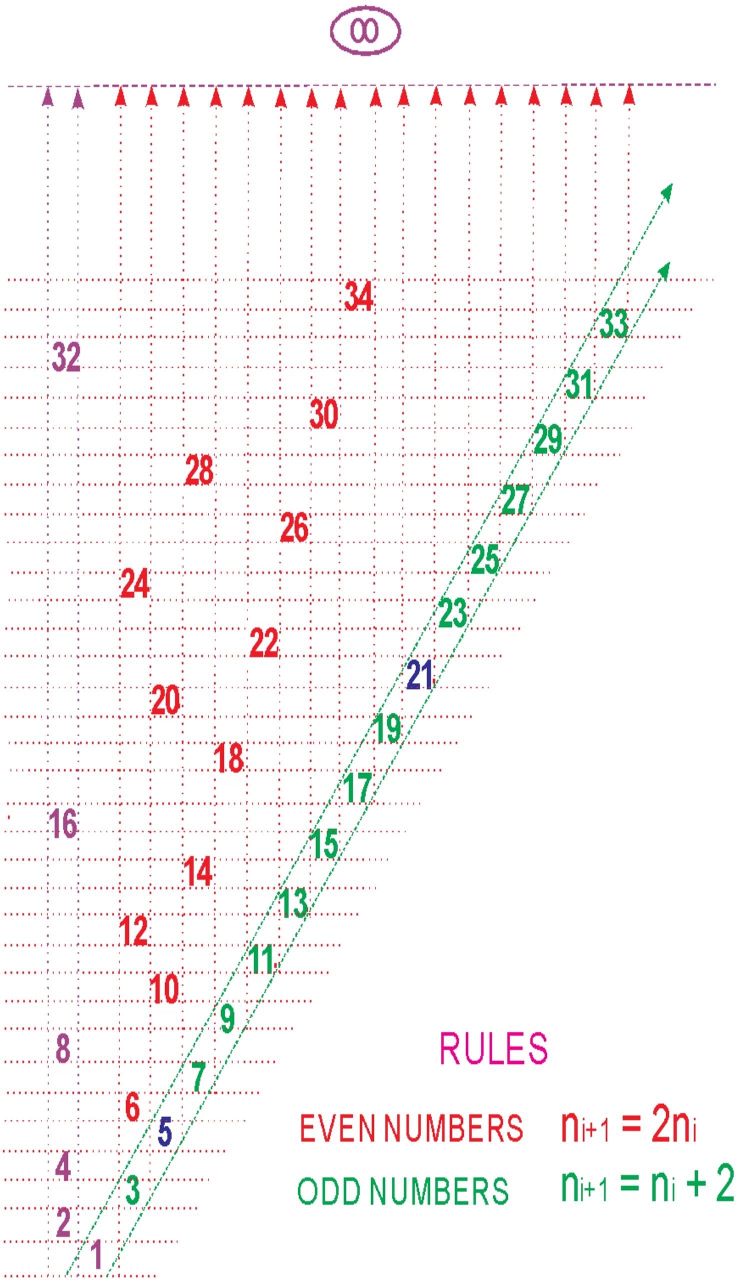
Figure 2 illustrates the Arrangement. The Main Even Sequence goes from "1" to infinity if we simply double each term to find the subsequent term of the geometric progression. Clearly, we will return from infinity to number "1" when following the inverse of said common ratio, in other words by dividing each term in the Main Even Sequence (including a theoretical number in infinity) by "2". It means that all Sequences that starts with a number in the Main Even Sequence only needs the rule "n/2" to reach number "1".

The rule "3n + 1" of the Conjecture is an additional arithmetic rule that creates a sophisticate means to make any other Sequence ruled by any odd number different from "1" to reach number "1" under the rule "n/2", in the same way it happens to all even numbers in the Main Even Sequence. Said sophisticate arithmetic mechanism is the supposed secret hidden in the Conjecture.

Based on the clear indication that the Main Even Sequence (in purple) is the key element to explain and prove the Conjecture, I used it as a model to create the Arrangement. I built the other Even Sequences (in red) from any other odd number from number "3" to infinity (in blue and green) by using the same common ratio "2", by successively multiplying each subsequent number by "2". As we see, each odd integer, from "1" to infinity, rules a unique and different Even Sequence (and performs as a "ruling number").

Figure 3 has the purpose to illustrate the Arrangement and show that it contains all integers from “1” to infinity, without repetition of numbers. I will use some alternative forms to represent the Arrangement, which will allow us to explain how the Sequences behave. All Even Sequences start with an odd number and reaches infinity under a geometric progression of even numbers and common ratio “2” ($n_{i+1} = 2n_i$). The Odd Sequence starts with number “1” and reaches infinity under an arithmetic progression of a common increment “2” ($n_{i+1} = n_i + 2$).

Figure 2: Even Sequences in the Arrangement



As we see in the partial representation in Table I, all other Even Sequences also comprise a geometric progression of a common ratio “4”. With the exception of the Even Sequences ruled by number “3” and the odd integers multiples of “3”, these geometric progressions in alternate order have an infinite quantity of terms (red numbers) that yield odd integers under the inverse of the rule “3n + 1”.

As a simple remark to clarify the use of the Arrangement, if we move from an odd number to infinity we deal with geometric progressions of a common ratio “4” or “2”. If we move backwards, the ratios are “1/4” and “1/2”, respectively.

Table I: Terms of Even Sequences

Even Sequences – ruling numbers and even terms											
Order of even terms	Ruling numbers $n_{i+1} = n_i + 2$										
	1	3	5	7	9	11		19	21	23	...
	Even terms $n_{i+1} = 2n_i$										
1st	2	6	10	14	18	22	...	38	42	46	...
2nd	4	12	20	28	36	44	...	76	84	92	...
3rd	8	24	40	56	72	88	...	152	168	184	...
4th	16	48	80	112	144	176	...	304	336	368	...
5th	32	96	160	224	288	352	...	608	672	736	...
6th	64	192	320	448	576	704	...	1216	1344	1472	...
...	...	...	...	...	...	...	...	...	...	...	...

In all Even Sequences, the terms of the referred geometric progression of a common ratio “4” (N_i) yield odd integers (n_i) under the inverse rule “$n_i = (N_i - 1)/3$”. These integers are Main Gates, directly connected to the Main Even Sequence, or Secondary Gates, connected to an Even Sequence ruled by a Main Gate or to an Even Sequence ruled by another Secondary Gate. The Main Gates and the Secondary Gates that transfer Sequences from an Even Sequence to another Even Sequence follow the relationship “$n_{i+1} = 4n_i + 1$”, as follows:

Even Sequence $N_{i+1} = 4N_i$

$$3n_{i+1} + 1 = 4(3n_i + 1)$$

Gates $n_{i+1} = 4n_i + 1$

The relationship above allows us to know all Main Gates and Secondary Gates from number “1” to infinity.

As the rationale to build an infinite sequence of numbers, we need to recall that if an arithmetic operation over a finite integer takes that number to infinity, the inverse operation must regenerate the starting number, whatever the meaning of infinity.

Conjecture rules and their inverses

The Arrangement in Figure 2 shows the Main Even Sequence that starts with number "1" (purple number), the Even Sequences that start with Main Gates (blue numbers), and the Even Sequences that start with Secondary Gates (green numbers).

Each Even Sequence is a geometric progression of a common ratio "2", which starts with an odd integer (from "1" to infinity) and reaches infinity. Except for each specific odd number that rules each Even Sequence, all Even Sequences only contain even numbers. I will show this is a relevant aspect to introduce the concept of "Reduced Sequence".

The Conjecture states that any Sequence of integers, whatever the initial number, will converge to number "1", if we follow two arithmetic rules: in case of an even number, divide it by "2"; in case of an odd number, multiply it by 3 and add "1". If we have a true statement, arithmetic requires that we may reach any elected number if we start with number "1" and move under the inverses of these two rules. In other words, we must have a reversible process. I used the inverses of the rules as follows:

In case of an odd number:

> We only apply the inverse rule "$n_{i+1} = 2n_i$", because, under the Conjecture rules, an odd number only results from an even number divided by "2".

In case of an even number:

> We apply the inverses of both rules: "$n_{i+1} = 2n_i$" and "$n_{i+1} = (N_i - 1)/3$", because, under the Conjecture rules, an even number may result from an odd number (under the rule "$n_{i+1} = 3n_i + 1$"), as well as from another even number (under the rule "$n_{i+1} = n_i/2$").

Then, if we take number "1" as the initial number, and follow the referred inverses of the Conjecture rules, we will be able to reach any even or odd integer. Figure 3 is a partial representation of the Arrangement in a different shape, with the purpose to illustrate the foregoing statement (brown arrows). The illustration makes clear the importance of the Main Even Sequence.

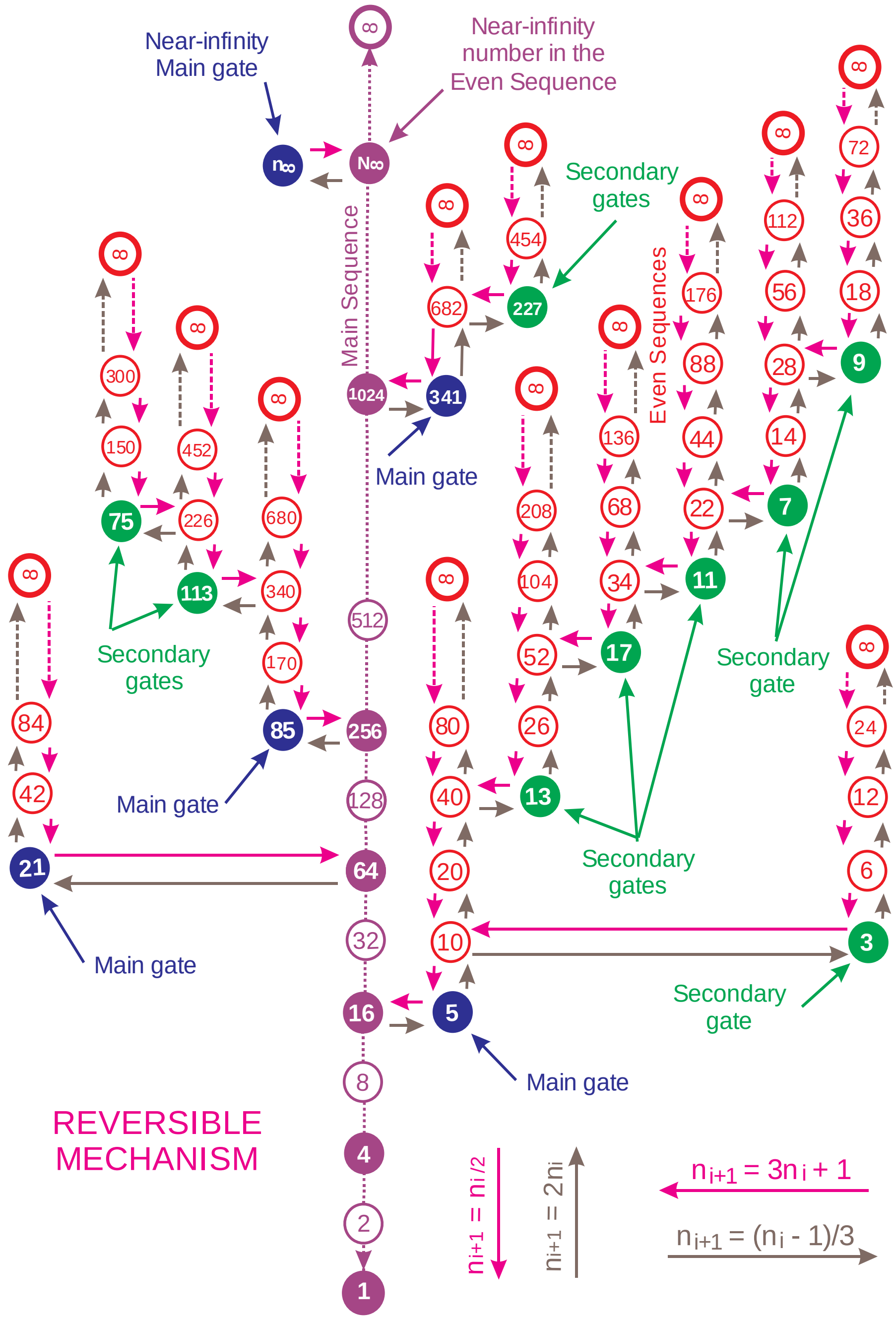
We built the alternative form of the Arrangement in Figure 3 from the powers of number "4" belonging to the Main Even Sequence ($N = 4^m$) and the inverses of the rules imposed by the Conjecture ($n_i = (N_i - 1)/3$ and $n_{i+1} = 2n_i$). As previously stated, in the Main Even Sequence only the powers of number "4" yield odd integers under the inverse of the rule "3n + 1".

The illustration shows some of the Secondary Gates (in green) and some of the Main Gates (in blue), in their respective roles to connect one Even Sequence to a different Even Sequence, as well as an Even Sequence to the Main Even Sequence (sections of geometric progressions).

By using the inverses of the rules required by the Conjecture, it is possible to follow an infinite ways to reach infinity from the powers of number "4" in the Main Even

Sequence. This occurs because when we find certain even numbers (as “40”, “52”, “34” and many others) there happens the option to apply each one of the inverses of the rules of the Conjecture, “$n_{i+1} = 2n_i$” or “$n_{i+1} = (N_i - 1)/3$” (the referred brown arrows).

Figure 3: Conjecture rules and their inverses



However, when we apply the Conjecture rules to the referred even numbers (“40”, “52”, “34” and many others) as the initial or as an intermediate number in any Sequence, there is no such option. It is mandatory to only apply the rule “$n_{i+1} = n_i/2$” to even integers, and the Sequence then created will have a unique path to follow, which brings it to the same power of “4” that originates the initial number. Figure 3 shows that restricted behavior (pink arrows). In brief, the Conjecture obeys a reversible process.

Clearly, if we move under the Conjecture rules, we will have an infinite number of infinite Sequences moving from infinity (each one with an infinite number of subsequences) towards number “1”.

The two rules of the Conjecture generate Sequences, which obey a reversible process. Each Sequence may go from number “1” to infinity, as well as come from infinity to number “1” following the same path in opposite directions, whatever the concept of infinity.

Ruling numbers and transfer gates

As illustrated in the Arrangement, each odd number rules an Even Sequence. Number “1” rules the Main Even Sequence, the Main Gates rule the Even Sequences directly connected to the Main Even Sequence, while the Secondary Gates rule the other Even Sequences. In the same way the Main Gates connect Sequences to the Main Even Sequence, the Secondary Gates connect Sequences to Even Sequences ruled by Main Gates or to Even Sequences ruled by other Secondary Gates.

When the Main Gates and Secondary Gates rule their respective Even Sequences, I refer to them as “ruling numbers”. When they transfer Sequences to other Even Sequences, I refer to them as “transfer gates”. Obviously, a same odd number may perform as a ruling number of an Even Sequence (from “1” to infinity), as well as a transfer gate in relation to another Even Sequence (from “3” to infinity).

The Even Sequences ruled by number “3” and the other odd numbers multiples of “3” (9, 15, 21, …) do not have transfer gates under the rule “3n + 1”. These numbers rule single Even Sequences, but they perform as transfer gates to other Even Sequences.

Each transfer gate under the rule “3n + 1” delivers an infinite quantity of Sequences to a specific Even Sequence, while the other rule “n/2” allows these Sequences to move downwards along the referred Even Sequence until it reaches its ruling number. Said ordered effect of the two rules creates an arithmetic mechanism that forces all Sequences to converge to number “1”.

As a result, the series of Main Gates (from “5” to infinity) following the relationship “$n_{i+1} = 4n_i + 1$”, as transfer gates, delivers an infinite number of Sequences to the Main Even Sequence under the rule “3n + 1”, and generates geometric progressions of a common ratio “4” in the Main Even Sequence.

In the same function of transfer gates, each series of Secondary Gates (from “3” to infinity) following the same relationship “$n_{i+1} = 4n_i + 1$” delivers an infinite number of Sequences to other Even Sequences, under the same rule “3n + 1”, and generates geometric progressions of a common ratio “4” in these Even Sequences.

Each odd number is the ruling odd number of a specific Even Sequence and the transfer gate of the Sequences comprised by said Even Sequence to a subsequent Even Sequence. As transfer gates, the odd numbers (Main Gates and Secondary Gates) account for the convergence of all Sequences to number “1”.

Types of Even Sequences

The previous concepts of ruling numbers and transfer gates allow us to classify the infinite quantity of Even Sequences in accordance with their structures, roles, ruling number and transfer gates. We see in Figures 4(a) to 4(c) three main types of Even Sequences.

Figure 4(a) represents the unique Main Even Sequence ruled by number "1" (in purple). The infinite number of Main Gates (in blue) are the transfer gates of said Even Sequence. The Main Even Sequence takes all Sequences to number "1" under the rule "n/2".

Figure 4(a): Main Even Sequence

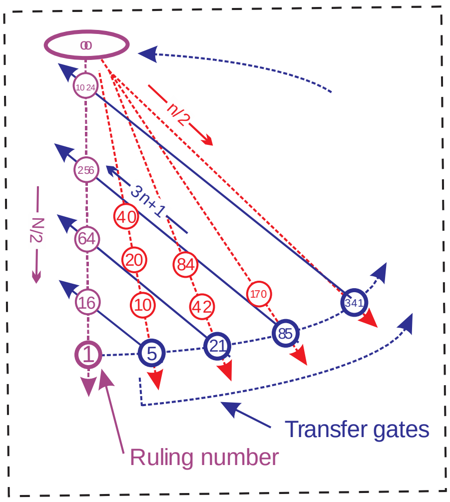


Figure 4(b) represents the Even Sequence ruled by number "5", a Main Gate (in blue). There are an infinite number of Main Gates, and each Main Gate receives an infinite number of transfer gates (in green). The Even Sequence ruled by a Main Gate goes to the Main Even Sequence under the rule "3n + 1".

Figure 4(b): Even Sequence ruled by a Main Gate

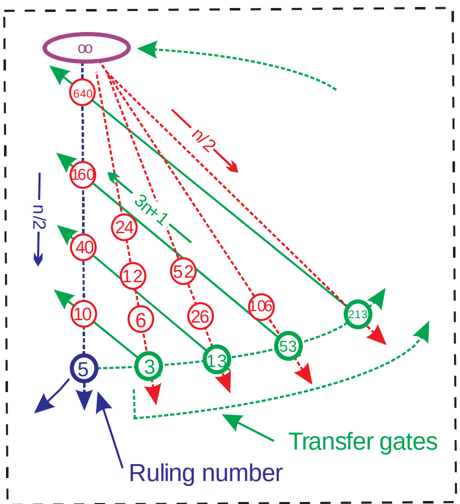
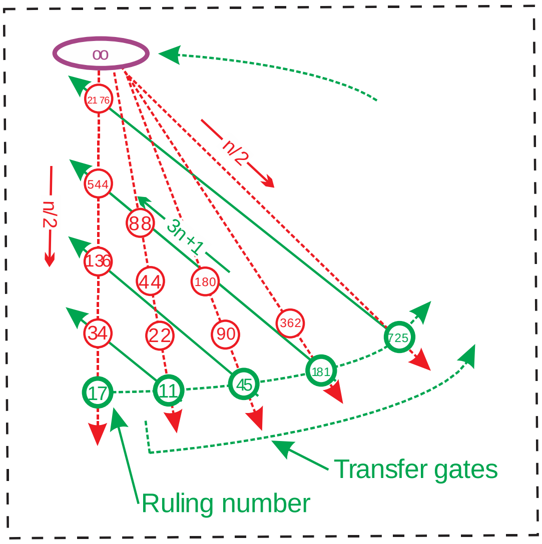


Figure 4(c) represents the Even Sequence ruled by number "17", a Secondary Gate (in green). There are an infinite number of Secondary Gates, and each Secondary Gate receives an infinite number of transfer gates (in green). The Even Sequence ruled by a Secondary Gate may go to another Even Sequence ruled by a Secondary Gate or to an Even Sequence ruled by a Main Gate under the rule "3n + 1".

Figure 4(c): Even Sequence ruled by a Secondary Gate



These three main types of Even Sequences have the same structure and work in the same manner, but with different roles. Besides, all of them have transfer gates, which follow the relationship "$n_{i+1} = 4n_i + 1$". Each Even Sequence has an infinite number of even numbers, which move from infinity to its ruling odd number under the rule "n/2", and receives Sequences from a series of different odd numbers, its transfer gates.

In addition to the three main types of Even Sequences above, there are three complementary types of Even Sequences ruled by number "3" and the odd multiples of "3" (as "9, 15, 21, …"). They are single Even Sequences because they do not receive Sequences from any transfer gate under the rule "3n + 1". However, they perform as transfer gates to other Even Sequences.

As illustrated in Figure 5, some of the Sequences from these single Even Sequences go directly to the Main Even Sequence (as the one ruled by "21"). Others go directly to an Even Sequence ruled by a Main Gate (as the one ruled by "3"). Finally, the remaining ones go to another Secondary Gate (as the one ruled by "69"). Nevertheless, their ruling Secondary Gates (odd numbers multiples of "3") yield geometric progressions of a common ratio "4" in other Even Sequences, and all the Sequences from these three complementary types of Even Sequences end in number "1". Obviously, each one of these Sequences comprises an infinite number of subsequences, when the initial number is a finite number.

I built the Arrangement by successively multiplying an odd number by "2" until infinity, what allows us to overcome the unknown meaning of infinity, since it works whatever infinity means. Under this building process of the Arrangement, arithmetic consistency requires infinity to be a number, whatever its size, but still a number. Besides, this unique Arrangement contains all odd and even numbers from “1” to infinity without repetition of any number.

The three complementary types of Even Sequences simply join the main types.

Convergence mechanism

Surprisingly, the reason behind the convergence of Sequences to number “1” is rather trivial. Each Sequence, whatever its starting number, freely moves through different geometric progressions until it reaches a geometric progression in the Main Even Sequence and ends in the loop “1, 4, 2, 1”.

The convergence of the Sequences occurs under the proper use of both rules. A Main Gate, as a transfer gate, freely delivers all the infinite quantity of Sequences it receives to a geometric progression of a common ratio “1/4” in the Main Even Sequence under the rule “3n + 1” (powers of number “4”). Similarly, a Secondary Gate, also as a transfer gate, freely delivers all the infinite quantity of Sequences it receives to different geometric progressions of a common ratio “1/4” in other Even Sequence, under the same rule. In each geometric progression, the rule “n/2” moves the Sequences downwards until they reach number “1”.

As a general behavior, the convergence occurs in three stages, as we see in Figure 5 for the specific case of an infinite Sequence (and its subsequences) ruled by the Secondary Gate ”7”. We have to keep in mind that, although starting with the referred Sequence, said Sequence (as all others) already carries an infinite number of previous Sequences it receives under the rule “3n + 1”, indicated in Figure 5 by the colored horizontal arrows, in addition to the subsequences it generates under the rule “n/2”.

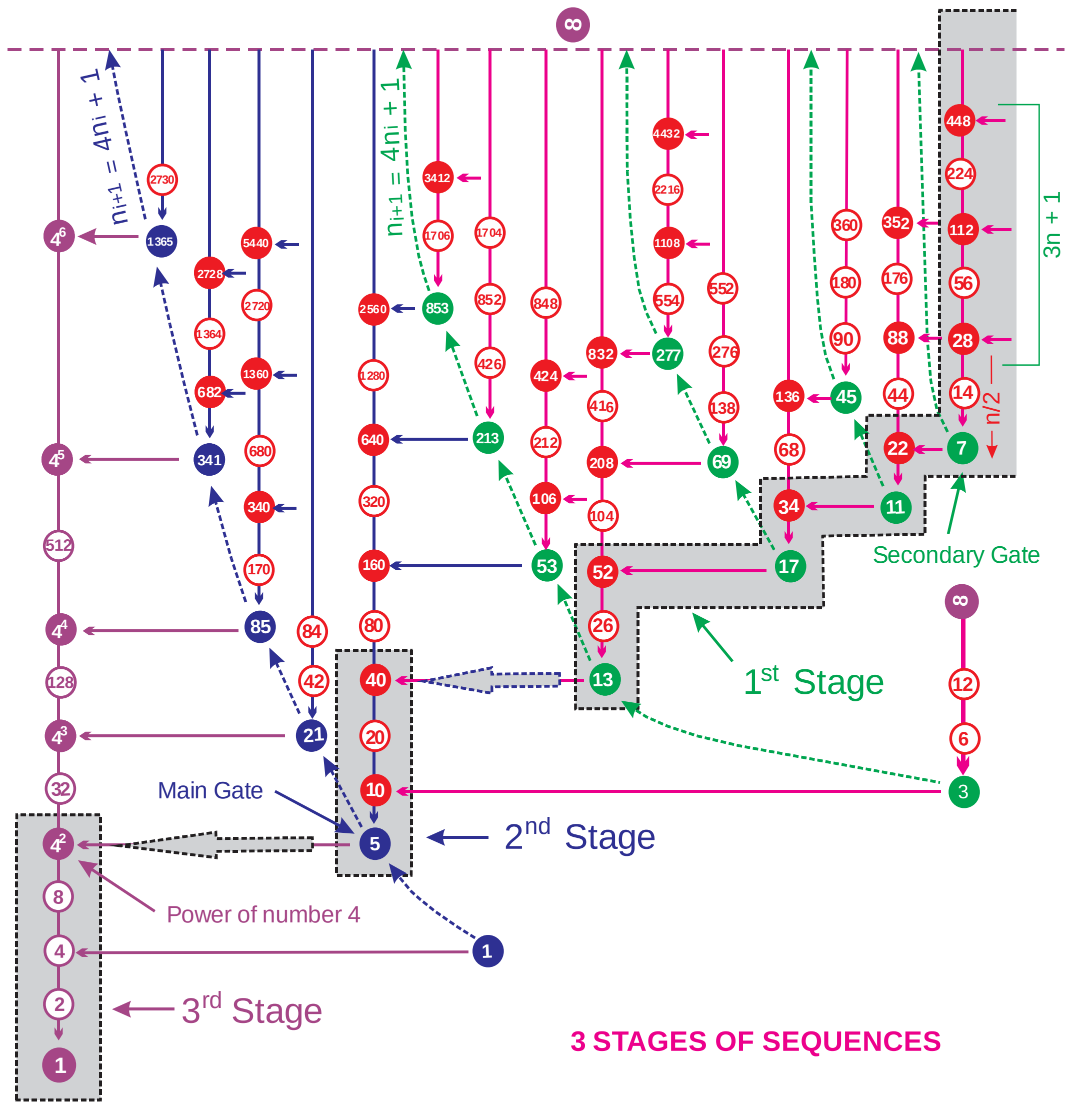
The first stage of any Sequence begins in infinity (or any finite number below infinity) and ends when that Sequence reaches the last Even Sequence ruled by a Secondary Gate. The second stage comprises a unique section of a geometric progression in an Even Sequence ruled by a Main Gate, which receives the Sequences delivered by the Secondary Gate of that last Even Sequence in the first stage. That Main Gate takes the Sequences to the Main Even Sequence under the rule “3n + 1. For each Sequence, the third and final stage is a unique section of the geometric progression in the Main Even Sequence (powers of number”4”) and the rule “n/2”.

We see that, for any specific Sequence, the first stage may include any number of sections of different geometric progressions and ends in a Secondary Gate (in green). The second stage is a unique section of a geometric progression and ends in a Main Gate (in blue). The third stage also comprises a unique section of a geometric progression and ends in number “1” (in purple). In any stage, a section used by a specific Sequence may comprise one or more terms of the relevant geometric progression.

A starting number in the Main Even Sequence only generates the third stage, while a starting number in an Even Sequence ruled by a Main Gate only exhibits the second and the third stages.

As it occurs with the Main Gate "5", the sections of geometric progressions of each other Main Gate are the second stages for an infinite number of Sequences. The sections of the geometric progression in the Main Even Sequence represent the third and final stage for all Sequences.

Figure 5: Convergence mechanism



For practical reasons, Figure 5 is partial view of the convergence mechanism, and shows the first stage exclusively for the Even Sequence ruled by the Secondary Gate "7". Said Secondary Gate receives its own infinite Sequence (and all of its subsequences) under the rule "n/2", as well as an infinite number of Sequences through its even numbers "28", "112", "448" and others under the rule "3n + 1".

All Sequences that reach the Secondary Gate "7" move to number "22" in the Even Sequence ruled by the Secondary Gate "11" under the rule "3n + 1". These Sequences from the Secondary Gate "7" join another group of an infinite number of Sequences, which enter the Even Sequence ruled by the Secondary Gate "11" through the other even numbers in that host Even Sequence, as "88', "352" and many others. The other Sequences received by the Secondary Gate "11" under the rule "n/2", including the

Sequences received under the rule “3n + 1” (pink arrows) are not included in the first stage of the Even Sequence ruled by number “7”. Clearly, each other Secondary Gate generates a different first stage, which may reach the Main Even Sequence through a different Main Gate, but the behavior will be the same.

Figure 5 shows that the relationship “$n_{i+1} = 4n_i + 1$” rules the series of Main Gates, as well as each series of Secondary Gates when they function as transfer gates. Each group of odd numbers that follow said relationship yields even numbers, which form geometric progressions of a common rate “4” in a same Even Sequence, what characterizes the convergence mechanism. The referred relationship permits to know all infinite transfer gates of any Even Sequence.

Similarly, all Sequences that reach the Secondary Gate “11” move to the Even Sequence ruled by the Secondary Gate “17” through the even number “34”. The Secondary Gate “17” takes the Sequences to the even number “52”, in the Even Sequence ruled by the Secondary Gate “13”. In our example, the first stage from the Even Sequence ruled by number “7” ends when the Sequences previously gathered reach the Even Sequence ruled by that Secondary Gate “13”, since the Main Gate “5” rules the next Even Sequence. In all Even Sequences these Sequences moves downwards with the help of the rule “n/2”.

In this particular example, the second stage comprises a section of the Even Sequence ruled by the Main Gate “5”, which receives the Sequences delivery by the Even Sequence ruled by the Secondary Gate “7”. In fact, all Sequences that reach the Secondary Gate “13”, and many others, reach the Main Gate “5”, as indicated in Figure 5 by pink arrows.

The Sequences move along sections of different geometric progressions. The rule “3n + 1” connects two consecutive sections of geometric progressions of a common ratio “1/4”, while the rule “n/2” modifies that common ratio to “1/2” The combined effect of the two rules forces the Sequences to reach number “1”.

As an example of sections of geometric progressions in a Sequence, let us consider a Sequence that starts with number 360 (Table II). We see from Figure 5 that said Sequence is indeed a subsequence of the infinite Sequence ruled by the Secondary Gate “45”, which belongs to a single Even Sequence (an Even Sequence with no transfer gates because “45” is an odd number multiple of “3”). We also see from Table II that in each different geometric progression, the relevant section may encompass a different number of terms. In red, we see the even numbers, which yield odd integers under the inverse of the rule “3n + 1”.

In Figure 5, a simplified illustration of the three stages of any Sequence, we only considered the Even Sequence ruled by the Secondary Gate “7”, and the previous Sequences it receives and passes ahead. The alternative form of the Arrangement presented in Figure 5 is a clear arithmetic representation of the Conjecture, and shows that every Sequence built under the two Conjecture rules, whatever the starting number, is simply a chain of sections of different geometric progressions that ends in number “1”.

A power of number "4" in the Main Even Sequence may receive only one infinite Sequence (single Even Sequences) or an infinite quantity of infinite Sequences (other Even Sequences). The overall behavior above discussed for the Sequences ruled by the Secondary Gate "7" will apply to all Sequences ruled by other Secondary Gates accordingly. It does not matter how many steps the iterative process may take, all Sequences reach a Main Gate and converge to number "1"., because there is no way to prevent a Sequence from leaping from an Even Sequence into a different Even Sequence, other than the final loop "1, 4, 2, 1".

Table II: Stages and sections of geometric progressions in a Sequence

Sequence stages	Sections of Geometric Progressions	Terms of Geometric Progressions (n/2)				Leap (3n + 1)
		Even numbers			Gates	
First stage	1st Section	360	180	90	45	136
	2nd Section	136	68	34	17	52
	3rd Section	52	26		13	40
Second stage	4th Section	40	20	10	5	16
Third stage	5th Section	16	8	4	1	4

Infinite Sequences and their standard path

The Arrangement shows that each odd number, from "1" to infinity, rules a unique individual Even Sequence. It means there are an infinite number of individual Sequences (geometric progressions of a common ratio "1/2") coming from infinity to their specific ruling odd number (a Main Gate, a Secondary Gate or number "1"). I refer to these Sequences as "infinite Sequences". Clearly, an infinite Sequence is a theoretical concept, which comprises an infinite number of subsequences, when the starting number is any number below infinity (a finite integer). Infinity is a common initial reference for all Sequences (in fact, a very large number).

In spite of sharing certain sequences of integers with other infinite Sequences in its path to number "1", each infinite Sequence remains as an independent Sequence until it reaches number "1".

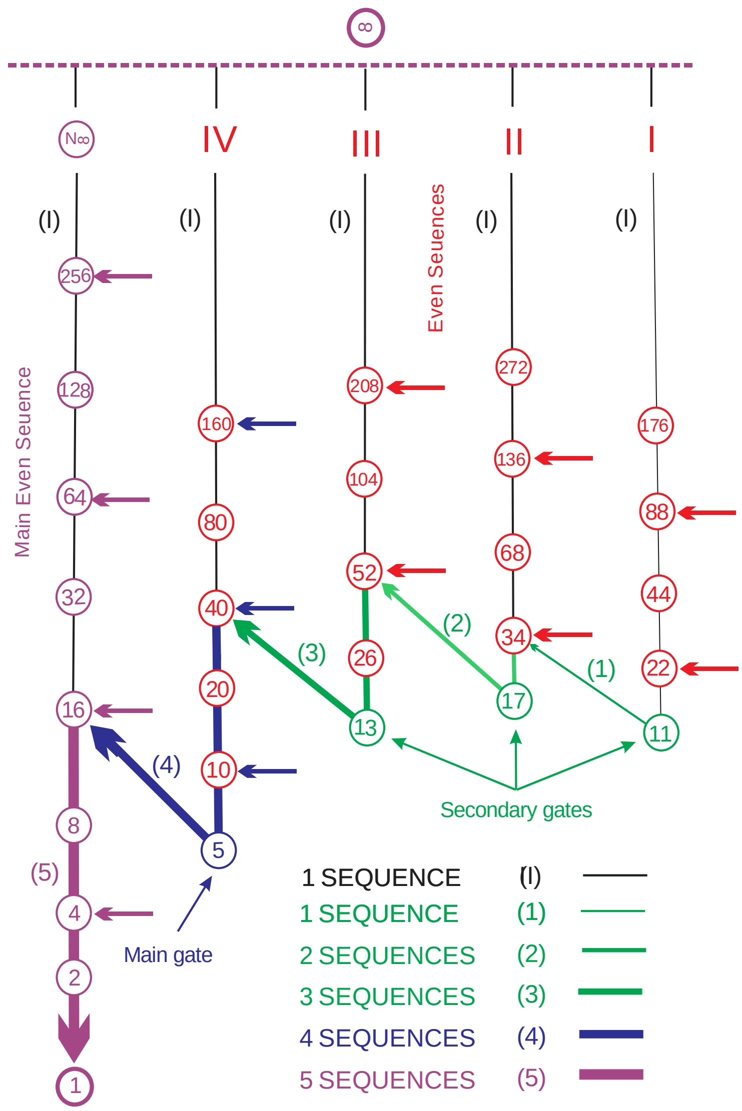
We may compare the Conjecture to a hydrographic basin. The initial number of a Sequence behaves like a water molecule, which departs from a water spring in a specific place and moves through many watercourses, sub-tributaries, tributaries and a main river until it reaches the ocean. In its path to the ocean, that molecule shares the same way with an infinite quantity of other molecules from different sources, but it remains as the same independent molecule it was when it started flowing.

Having in mind that an infinite Sequence freely moves from an Even Sequence to a different Even Sequence, in its way to number "1" it may overlap a certain subsequence, which initiates in and belongs to another Even Sequence as its own subsequence without losing its individuality. Each infinite Sequence has its own standard path, as in Figure 6.

As we see in Figure 6, a Sequence that starts with number “52” is indeed a subsequence of the infinite Sequence ruled by the Secondary Gate “13”. Nevertheless, said Sequence also is a subsequence of other infinite Sequences ruled by other Secondary Gates, as “17”, “11” and many others.

Figure 6 does not contemplate the Sequences carried by each infinite Sequence, delivered to Secondary Gates (red arrows), delivered to Main Gates (blue arrows), and delivered to the Main Even Sequence (purple arrows). Since we did not consider the Sequences delivered by the transfer gates, we started with only one infinite Sequence (ruled by the Secondary Gate “11”) and ended up with just five ones. The example in Figure 6 left out an infinite quantity of Sequences.

Figure 6: Standard path



The Sequences created under the rules of the Conjecture do not go to infinity. They come from infinity. Besides, they do not have a chaotic and unpredictable nature.

GSJ: Volume 13, Issue 1, January 2025
ISSN 2320-9186

Contrarily, they obey an ingenious arithmetic mechanism, which forces them to converge to number "1", after reaching a power of number "4" in the Main Even Sequence.

The Conjecture deals with infinity and even the most powerful supercomputer cannot prove it numerically. To be consistent, we have to accept infinity as a very large number, but still a number and that the Sequences will behave in the same way they do in the realm of finite numbers. Otherwise, the Conjecture has no arithmetic meaning. My conceptual approach with the help of the Arrangement allows us to overcome this difficulty because the convergence mechanism must work for any starting number, whatever the meaning of infinity, which will mandatorily be a number.

Reduced Sequence

Each infinite Sequence is in fact a chain of sections of different geometric progressions of even integers, which starts in infinity and progresses under a common ratio of "1/2" until it reaches a first odd integer (its ruling number, usually a Secondary Gate "n_i"). Then, any finite number "N_i" in any Sequence, no matter how large it may be, is a multiple of "2" in respect of that ruling number, as follows:

$N_i = 2^m(n_i)$ m = an integer

Clearly,

$n_i = N_i/2^m$

It means we can ignore the intermediate even numbers in between the initial finite number "N_i" and the respective ruling number "n_i" of any infinite Sequence, in a manner to reach that ruling number in a single step. In case of the Main Even Sequence, the Sequence goes directly to number "1". In some other cases, the Sequence goes directly to a Main Gate, and then to number "1". In the other cases, the Sequences alternates even and odd integers until they meet a Main Gate. I refer to this alternative form of Sequences as "Reduced Sequences", which may be mathematically expressed as follows:

Start with odd integer «O_i»

$$E_i = 3O_i + 1 \rightarrow O_{i+1} = E_i/2^{m_1} \rightarrow E_{i+1} = 3O_{i+1} + 1 \rightarrow O_{i+2} = E_{i+1}/2^{m_2}$$

Start with even integer «E_i»

O_i = an odd integer

E_i = an even integer

m_i = an integer, equal to or greater than "1"

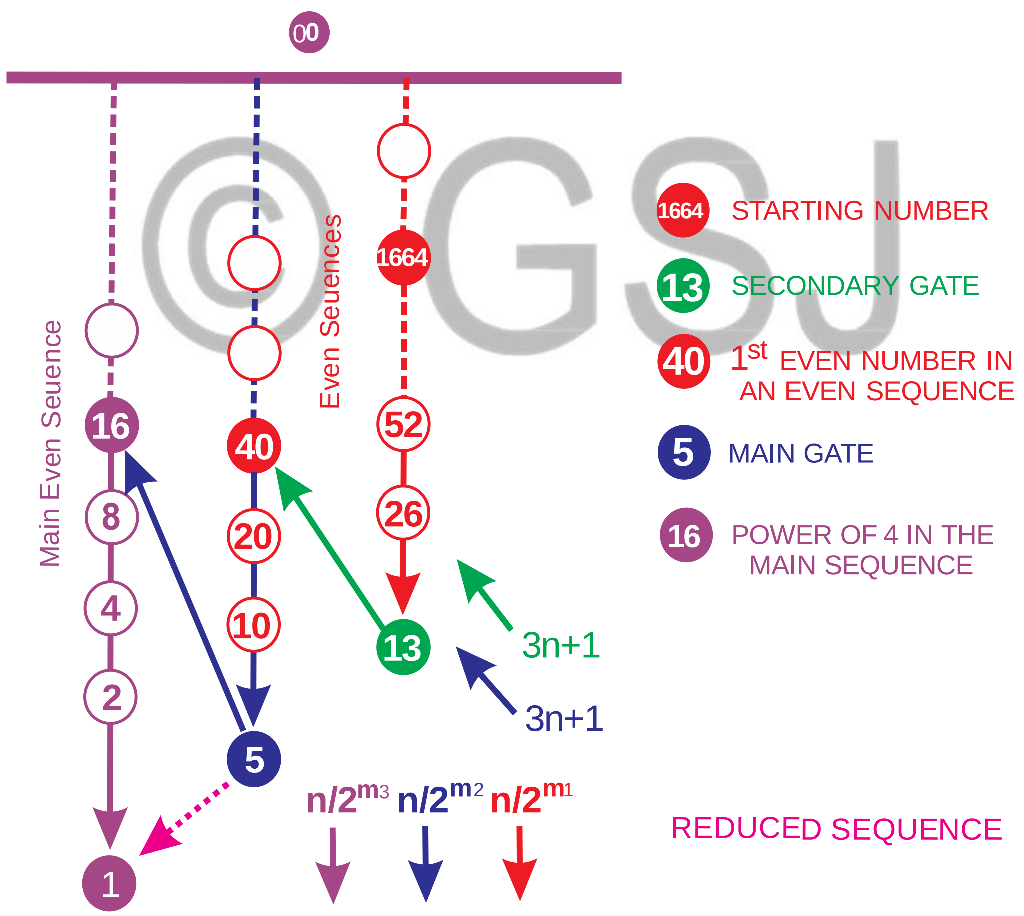
The convergence formula indicates that if the Sequence starts with an odd number, the rule “3n + 1” immediately takes it to an even number, what validates the concept of Reduced Sequences.

Let us assume we want the Reduced Sequence by starting with number “1,664”. We find the ruling odd number (Secondary Gate) of the Even Sequence as “$1{,}664/2^7 = 13$”. There is a unique odd number for each Even Sequence. If we use a divider and obtain another even number, it means we are in the same Even Sequence and need a greater divider. If we use a divider and obtain a fractional number, it means we extrapolated the Even Sequence and need a lower divider.

I do not know if any other author has already presented that concept of Reduced Sequence, particularly under the rationale of my approach.

Figure 6 illustrates an example of a Reduced Sequence when we start with number “1,664”. The Reduced Sequence is “1,664, 13, 40, 5, 16, 1” (only five steps).

Figure 6: Reduced Sequence



It is possible to elaborate a computer program and determine the alternate numbers (even and odd integers) of the Reduced Sequence of any Sequence, by avoiding the meaningless intermediate even numbers and minimizing the number of steps in any Sequence. Alternatively, we may use one of the programs available on the internet (as Python-based algorithms), which yields the entire Sequence for any number, and only consider the Reduced Sequence by ignoring the intermediate even numbers. The math expression of a Reduced Sequence indicates that a Sequence is never a number arrangement of a chaotic and unpredictable nature. Contrarily, it shows that all Sequences follow a pre-determined path and converge to number "1".

As an example, if we initiate a Sequence with number "86,016", we would face a great number of steps until number "1" (18 steps). Having in mind that "86,016/2^{12} = 21", the Reduced Sequence will only need three steps. The number "348,160" is another example, when three steps suffice.

86,016	21	64	1	(3 steps x 18 steps)
348,160	85	256	1	(3 steps x 22 steps)

Any Sequence may freely move from an Even Sequence to another Even Sequence, meeting different even numbers and different odd numbers, and all of them, no matter how many steps it takes, will join the Main Even Sequence, and will end in number "1".

An Even Sequence ruled by a specific Secondary Gate comprises a certain infinite Sequence. Another odd number that (as a transfer gate) delivers Sequences to said Even Sequence is the ruling number (Secondary Gate) of a different Even Sequence, which comprises a different infinite Sequence. Since the Arrangement does not contemplate repetition of any number, this condition prevents that certain infinite Sequence to return to its Even Sequence and face a loop other than the final one, "1, 4, 2, 1".

The Arrangement also clarifies the reason why any Sequence contains more even numbers than odd numbers when following the two Conjecture rules. Any time an odd number occurs in a Sequence, the rule "3n + 1" imposes a new even number, but an even number may lead to a sequence of even numbers before an odd number occurs again.

However, when using the Reduced Sequence, there will occur the same quantity of even and odd numbers, if the Sequence starts with an even number, and only one extra odd number, in case the Sequence starts with an odd number.

Any Sequence is in fact a sequence of numbers, which alternates even and odd numbers until it reaches number "1".

Conclusions

Each Sequence, whatever the starting number, is a chain of sections of different geometric progressions of integers that ends in number "1". Each section moves downwards under the rule "n/2", and ends in an odd number, a Secondary Gate, a Main Gate or number "1". The rule "3n + 1" makes the connection between two consecutive sections by avoiding the fractional numbers, which would result from the application of the rule "n/2" to the odd number that ends the first or any intermediate section of the chain of sections of different geometric progressions of any Sequence.

The Conjecture rules imply a unique Arrangement, which comprises all even and odd integers from “1” to infinity without repetition of any number. Said Arrangement is the arithmetic representation of the Conjecture, which adequately used explains the puzzle. The rationale of the present approach to the Conjecture through such Arrangement allows us to overcome the difficulty of facing the concept of infinity.

In theory, the first section of any Sequence is a mother geometric progression, which begins in infinity (or any finite number below infinity) and ends in an odd number (in general a Secondary Gate). The last section begins in a power of number “4” in the Main Even Sequence and ends in number “1”. Each intermediate section begins with an even number in an Even Sequence and ends in a different odd number that rules the Even Sequence, either a Secondary Gate or a Main Gate. Before reaching the Main Even Sequence, every common Sequence passes through a Main Gate.

People interested in the Conjecture know it as an unsolvable or undecidable conundrum, which is out of reach of current math. We often see the Conjecture listed among the top ten unsolved mathematical problems. Not in line with such understanding, I say there is no mystery in the Conjecture, only elementary arithmetic. The Sequences are not arrangements of numbers of chaotic and unpredictable nature. Contrarily, they obey a creative arithmetic mechanism, which results from the combined effect of different geometric progressions of a common ratio “1/2” (the rule “n/2”) and a common ratio “1/4” (the rule “3n + 1”). The referred elementary arithmetic mechanism is the secret hidden in this famous math problem.

The Even Sequences ruled by number “3” and odd numbers multiples of number “3” are single Even Sequences, since they do not have transfer gates (no even integers results from the rule “3n + 1”). Nevertheless, number “3” and all these odd numbers multiples of number “3” work as transfer gates in respect of other Even Sequences.

It is also possible to represent any Sequence as an alternation of even and odd numbers, a simplified form that substantially reduces the number of steps needed to reach number “1” (Reduced Sequence), a concept that confirms the convergence of the Sequences.

In brief, whatever the starting number, and regardless of the number of steps it takes, each Sequence freely moves from an Even Sequence to a different Even Sequence under the rule "3n + 1". Within each Even Sequence, the Sequence moves downwards to the ruling number of said Even Sequence under the rule "n/2". The process goes on until the Sequence reaches the Main Even Sequence and converges to number "1".

As a final comment in this paper, I want to emphasize that, indeed, the Conjecture deals with subsequences, once we never reach infinity. Each infinite Sequence is a theoretical Sequence to represent all subsequences ruled by a same odd number, since once started they follow the same path until number "1".